

Supplementary Materials for “Generalized local Kendall’s τ : a novel framework for uncovering nonlinear local dependence”

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This Web Appendix consists of six sections. Appendix A provides proofs of all theorems and remarks. Appendix B introduces four strategies for defining local regions in detail, along with their applicable scenarios, and compares their advantages and limitations. Appendix C plots Type I and Type II local dependence surfaces for Clayton, Gumbel and Frank copula models. Appendix D reports the comparison results of global and local Kendall’s τ for Gumbel, Clayton, Frank, and FGM copula models. Appendix E presents additional simulation results, analyzes the underlying mechanisms for the superior performance of the copula-based estimator, and reports the results of the copula model misspecification experiments. Appendix F provides additional results of the two data examples and introduces the Archimedean mixture copula models employed.

Web Appendix A Proofs

A.1 Proof of Theorem 2

Let (X_1, Y_1) and (X_2, Y_2) be two independent realizations of (X, Y) , and let Ω_{12} denote the event $\{X_1, X_2 \in [X^l, X^u], Y_1, Y_2 \in [Y^l, Y^u]\}$. According to Equation (1), the population version of $\tau_{XY}^l(X^l, X^u, Y^l, Y^u)$ can be expressed as

$$\begin{aligned} & \tau_{XY}^l(X^l, X^u, Y^l, Y^u) \\ &= \mathbb{P}[(X_1 - X_2)(Y_1 - Y_2) > 0 \mid \Omega_{12}] - \mathbb{P}[(X_1 - X_2)(Y_1 - Y_2) < 0 \mid \Omega_{12}] \\ &= 2\mathbb{P}[(X_1 - X_2)(Y_1 - Y_2) > 0 \mid \Omega_{12}] - 1 \\ &= 2 \left\{ \frac{\mathbb{P}[F_X^{-1}(p^l) \leq X_1 < X_2 \leq F_X^{-1}(p^u), F_Y^{-1}(q^l) \leq Y_1 < Y_2 \leq F_Y^{-1}(q^u)]}{\mathbb{P}[F_X^{-1}(p^l) \leq X_1, X_2 \leq F_X^{-1}(p^u), F_Y^{-1}(q^l) \leq Y_1, Y_2 \leq F_Y^{-1}(q^u)]} \right. \\ & \quad \left. + \frac{\mathbb{P}[F_X^{-1}(p^l) \leq X_2 < X_1 \leq F_X^{-1}(p^u), F_Y^{-1}(q^l) \leq Y_2 < Y_1 \leq F_Y^{-1}(q^u)]}{\mathbb{P}[F_X^{-1}(p^l) \leq X_1, X_2 \leq F_X^{-1}(p^u), F_Y^{-1}(q^l) \leq Y_1, Y_2 \leq F_Y^{-1}(q^u)]} \right\} - 1. \quad (\text{S1}) \end{aligned}$$

We compute the two numerators in the above expression by integrating over the distribution of (X_1, Y_1) , and obtain

$$\begin{aligned} & \mathbb{P}[F_X^{-1}(p^l) \leq X_1 < X_2 \leq F_X^{-1}(p^u), F_Y^{-1}(q^l) \leq Y_1 < Y_2 \leq F_Y^{-1}(q^u)] \\ &= \int_{F_Y^{-1}(q^l)}^{F_Y^{-1}(q^u)} \int_{F_X^{-1}(p^l)}^{F_X^{-1}(p^u)} \mathbb{P}[F_X^{-1}(p^l) \leq x < X_2 \leq F_X^{-1}(p^u), \\ & \quad F_Y^{-1}(q^l) \leq y < Y_2 \leq F_Y^{-1}(q^u)] dC_1(F_X(x), F_Y(y)) \\ &= \int_{F_Y^{-1}(q^l)}^{F_Y^{-1}(q^u)} \int_{F_X^{-1}(p^l)}^{F_X^{-1}(p^u)} [C_2(p^u, q^u) - C_2(F_X(x), q^u) \\ & \quad - C_2(p^u, F_Y(y)) + C_2(F_X(x), F_Y(y))] dC_1(F_X(x), F_Y(y)), \quad (\text{S2}) \end{aligned}$$

and

$$\begin{aligned} & \mathbb{P}[F_X^{-1}(p^l) \leq X_2 < X_1 \leq F_X^{-1}(p^u), F_Y^{-1}(q^l) \leq Y_2 < Y_1 \leq F_Y^{-1}(q^u)] \\ &= \int_{F_Y^{-1}(q^l)}^{F_Y^{-1}(q^u)} \int_{F_X^{-1}(p^l)}^{F_X^{-1}(p^u)} \mathbb{P}[F_X^{-1}(p^l) \leq X_2 < x \leq F_X^{-1}(p^u), \\ & \quad F_Y^{-1}(q^l) \leq Y_2 < y \leq F_Y^{-1}(q^u)] dC_1(F_X(x), F_Y(y)) \\ &= \int_{F_Y^{-1}(q^l)}^{F_Y^{-1}(q^u)} \int_{F_X^{-1}(p^l)}^{F_X^{-1}(p^u)} [C_2(F_X(x), F_Y(y)) - C_2(p^l, F_Y(y)) \\ & \quad - C_2(F_X(x), q^l) + C_2(p^l, q^l)] dC_1(F_X(x), F_Y(y)). \quad (\text{S3}) \end{aligned}$$

Similarly, we compute the denominator of (S1), and get

$$\begin{aligned} & \mathbb{P}[F_X^{-1}(p^l) \leq X_1, X_2 \leq F_X^{-1}(p^u), F_Y^{-1}(q^l) \leq Y_1, Y_2 \leq F_Y^{-1}(q^u)] \\ &= \int_{F_Y^{-1}(q^l)}^{F_Y^{-1}(q^u)} \int_{F_X^{-1}(p^l)}^{F_X^{-1}(p^u)} [C_2(p^u, q^u) - C_2(p^l, q^u) - C_2(p^u, q^l) + C_2(p^l, q^l)] dC_1(F_X(x), F_Y(y)) \\ &= [C_2(p^u, q^u) - C_2(p^l, q^u) - C_2(p^u, q^l) + C_2(p^l, q^l)] [C_1(p^u, q^u) - C_1(p^l, q^u) \\ & \quad - C_1(p^u, q^l) + C_1(p^l, q^l)]. \quad (\text{S4}) \end{aligned}$$

Since $C_1 = C_2 = C$, after applying the probability integral transforms $U = F_X(x)$ and $V = F_Y(y)$, (S1) can be further expressed as

$$\begin{aligned}
\tau_{XY}^l(X^l, X^u, Y^l, Y^u) &= \tau_{UV}^l(p^l, p^u, q^l, q^u) \\
&= 2 \left(\frac{\int_{q^l}^{q^u} \int_{p^l}^{p^u} [C(p^u, q^u) - C(u, q^u) - C(p^u, v) + C(u, v)] dC(u, v)}{[C(p^u, q^u) - C(p^l, q^u) - C(p^u, q^l) + C(p^l, q^l)]^2} \right. \\
&\quad \left. + \frac{\int_{q^l}^{q^u} \int_{p^l}^{p^u} [C(u, v) - C(p^l, v) - C(u, q^l) + C(p^l, q^l)] dC(u, v)}{[C(p^u, q^u) - C(p^l, q^u) - C(p^u, q^l) + C(p^l, q^l)]^2} \right) - 1 \\
&= \frac{2 \int_{q^l}^{q^u} \int_{p^l}^{p^u} [2C(u, v) - C(u, q^l) - C(u, q^u) - C(p^l, v) - C(p^u, v) + C(p^l, q^l) + C(p^u, q^u)] dC(u, v)}{[C(p^u, q^u) - C(p^l, q^u) - C(p^u, q^l) + C(p^l, q^l)]^2} - 1. \quad (\text{S5})
\end{aligned}$$

Since

$$\int_{q^l}^{q^u} \int_{p^l}^{p^u} [C(p^l, q^l) + C(p^u, q^u)] dC(u, v) = [C(p^l, q^l) + C(p^u, q^u)] \frac{[C(p^u, q^u) - C(p^l, q^u) - C(p^u, q^l) + C(p^l, q^l)]}{-C(p^u, q^l) + C(p^l, q^l)},$$

$\tau_{XY}^l(X^l, X^u, Y^l, Y^u)$ can finally be simplified as

$$\begin{aligned}
\tau_{XY}^l(X^l, X^u, Y^l, Y^u) &= \frac{2 \int_{q^l}^{q^u} \int_{p^l}^{p^u} [2C(u, v) - C(u, q^l) - C(u, q^u) - C(p^l, v) - C(p^u, v)] dC(u, v)}{[C(p^u, q^u) - C(p^l, q^u) - C(p^u, q^l) + C(p^l, q^l)]^2} \\
&\quad + \frac{C(p^u, q^u) + C(p^l, q^u) + C(p^u, q^l) + C(p^l, q^l)}{C(p^u, q^u) - C(p^l, q^u) - C(p^u, q^l) + C(p^l, q^l)}. \quad (\text{S6})
\end{aligned}$$

Theorem 2 has been proved.

A.2 Proof of Remark 2

From (S6), we know that

$$\begin{aligned}
&\lim_{q^l \rightarrow 0} \lim_{q^u \rightarrow 1} \lim_{p^l \rightarrow 0} \lim_{p^u \rightarrow 1} \tau_{UV}^l(p^l, p^u, q^l, q^u) \\
&= \frac{2 \int_0^1 \int_0^1 [2C(u, v) - C(u, 0) - C(u, 1) - C(0, v) - C(1, v)] dC(u, v)}{[C(1, 1) - C(0, 1) - C(1, 0) + C(0, 0)]^2} \\
&\quad + \frac{C(1, 1) + C(0, 1) + C(1, 0) + C(0, 0)}{C(1, 1) - C(0, 1) - C(1, 0) + C(0, 0)} \\
&= 2 \int_0^1 \int_0^1 [2C(u, v) - u - v] dC(u, v) + 1.
\end{aligned}$$

Since $\mathbb{E}(U) = \mathbb{E}(V) = 1/2$ (Nelsen, 2006, p. 160), we have $\int_0^1 \int_0^1 u dC(u, v) = \int_0^1 \int_0^1 v dC(u, v) = \frac{1}{2}$. Therefore, $\lim_{q^l \rightarrow 0} \lim_{q^u \rightarrow 1} \lim_{p^l \rightarrow 0} \lim_{p^u \rightarrow 1} \tau_{UV}^l(p^l, p^u, q^l, q^u) = 4 \int_0^1 \int_0^1 C(u, v) dC(u, v) - 1$.

A.3 Proof of Theorem 3

According to Equations (5) and (6), we have

$$\begin{aligned}
\tau_{UV}^{UU}(p, q) &= \tau_{UV}^l(p, 1, q, 1) = \frac{2 \int_q^1 \int_p^1 [2C(u, v) - C(u, q) - C(p, v) - C(1, v)] dC(u, v)}{[C(1, 1) - C(p, 1) - C(1, q) + C(p, q)]^2} \\
&\quad + \frac{C(1, 1) + C(p, 1) + C(1, q) + C(p, q)}{C(1, 1) - C(p, 1) - C(1, q) + C(p, q)} \\
&= \frac{2 \int_q^1 \int_p^1 [2C(u, v) - C(u, q) - C(p, v) - u - v] dC(u, v)}{[1 - p - q + C(p, q)]^2} + \frac{1 + p + q + C(p, q)}{1 - p - q + C(p, q)}. \quad (S7)
\end{aligned}$$

Next,

$$\begin{aligned}
\tau_{UV}^{UL}(p, q) &= \tau_{UV}^l(p, 1, 0, q) = \frac{2 \int_0^q \int_p^1 [2C(u, v) - C(u, 0) - C(p, v) - C(1, v)] dC(u, v)}{[C(1, q) - C(p, q) - C(1, 0) + C(p, 0)]^2} \\
&\quad + \frac{C(1, q) + C(p, q) + C(1, 0) + C(p, 0)}{C(1, q) - C(p, q) - C(1, 0) + C(p, 0)} \\
&= \frac{2 \int_0^q \int_p^1 [2C(u, v) - C(u, q) - C(p, v) - v] dC(u, v)}{[q - C(p, q)]^2} + \frac{q + C(p, q)}{q - C(p, q)}. \quad (S8)
\end{aligned}$$

Similarly,

$$\tau_{UV}^{LU}(p, q) = \tau_{UV}^l(0, p, q, 1) = \frac{2 \int_q^1 \int_0^p [2C(u, v) - C(p, v) - C(u, q) - u] dC(u, v)}{[p - C(p, q)]^2} + \frac{p + C(p, q)}{p - C(p, q)}. \quad (S9)$$

Since

$$\begin{aligned}
\int_0^q \int_p^1 C(u, q) dC(u, v) &= \int_0^q \int_p^1 C(u, q) \frac{\partial C(u, v)}{\partial u \partial v} du dv = \int_p^1 C(u, q) \left[\int_0^q \frac{\partial C(u, v)}{\partial u \partial v} dv \right] du \\
&= \int_p^1 C(u, q) \left[\frac{\partial C(u, q)}{\partial u} - \frac{\partial C(u, 0)}{\partial u} \right] du = \int_p^1 C(u, q) dC(u, q) \\
&= \frac{1}{2} [C(1, q)^2 - C(p, q)^2] = \frac{1}{2} [q^2 - C(p, q)^2]
\end{aligned}$$

and

$$\begin{aligned}
\int_q^1 \int_0^p C(p, v) dC(u, v) &= \int_q^1 \int_0^p C(p, v) \frac{\partial C(u, v)}{\partial u \partial v} du dv = \int_q^1 C(p, v) \left[\int_0^p \frac{\partial C(u, v)}{\partial u \partial v} du \right] dv \\
&= \int_q^1 C(p, v) \left[\frac{\partial C(p, v)}{\partial v} - \frac{\partial C(0, v)}{\partial v} \right] dv = \int_q^1 C(p, v) dC(p, v) \\
&= \frac{1}{2} [C(p, 1)^2 - C(p, q)^2] = \frac{1}{2} [p^2 - C(p, q)^2],
\end{aligned}$$

(S8) and (S9) can be further simplified as

$$\tau_{UV}^{UL}(p, q) = \frac{2 \int_0^q \int_p^1 [2C(u, v) - C(p, v) - v] dC(u, v)}{[q - C(p, q)]^2} \quad (\text{S10})$$

and

$$\tau_{UV}^{LU}(p, q) = \frac{2 \int_q^1 \int_0^p [2C(u, v) - C(u, q) - u] dC(u, v)}{[p - C(p, q)]^2}, \quad (\text{S11})$$

respectively.

Finally,

$$\begin{aligned} \tau_{UV}^{LL}(p, q) &= \tau_{UV}^l(0, p, 0, q) \\ &= \frac{2 \int_0^q \int_0^p [2C(u, v) - C(u, 0) - C(u, q) - C(0, v) - C(p, v)] dC(u, v)}{[C(p, q) - C(0, q) - C(p, 0) + C(0, 0)]^2} \\ &\quad + \frac{C(p, q) + C(0, q) + C(p, 0) + C(0, 0)}{C(p, q) - C(0, q) - C(p, 0) + C(0, 0)} \\ &= \frac{2 \int_0^q \int_0^p [2C(u, v) - C(u, q) - C(p, v)] dC(u, v)}{C(p, q)^2} + 1. \end{aligned} \quad (\text{S12})$$

Since

$$\begin{aligned} \int_0^q \int_0^p C(p, v) dC(u, v) &= \int_0^q \int_0^p \left[C(p, v) \frac{\partial^2 C(u, v)}{\partial u \partial v} du \right] dv = \int_0^q C(p, v) \left[\int_0^p \frac{\partial^2 C(u, v)}{\partial u \partial v} du \right] dv \\ &= \int_0^q C(p, v) \left[\frac{\partial C(p, v)}{\partial v} - \frac{\partial C(0, v)}{\partial v} \right] dv \\ &= \int_0^q C(p, v) \frac{\partial C(p, v)}{\partial v} dv = \frac{1}{2} C(p, q)^2 \end{aligned}$$

and

$$\int_0^q \int_0^p C(u, q) dC(u, v) = \frac{1}{2} C(p, q)^2,$$

(S12) can be further simplified as

$$\tau_{UV}^{LL}(p, q) = \frac{2 \int_0^q \int_0^p 2C(u, v) dC(u, v) - 2C(p, q)^2}{C(p, q)^2} + 1 = \frac{4 \int_0^q \int_0^p C(u, v) dC(u, v)}{C(p, q)^2} - 1. \quad (\text{S13})$$

Theorem 3 has been proved.

A.4 Proof of Remark 4

As we know, $F_{-X}^{-1}(1-p) = -F_X^{-1}(p)$, therefore

$$\begin{aligned}\tau_{-X,Y}^{LL}(F_{-X}^{-1}(1-p), F_Y^{-1}(q)) &= \tau_{-X,Y}^{LL}(-F_X^{-1}(p), F_Y^{-1}(q)) = -\tau_{X,Y}^{UL}(F_X^{-1}(p), F_Y^{-1}(q)) \\ &= -\tau_{U,V}^{UL}(p, q).\end{aligned}$$

Since $\tau_{-X,Y}^{LL}(F_{-X}^{-1}(1-p), F_Y^{-1}(q)) = \tau_{1-U,V}^{LL}(1-p, q)$, we have $\tau_{U,V}^{UL}(p, q) = -\tau_{1-U,V}^{LL}(1-p, q)$. Similarly, we can prove $\tau_{U,V}^{LU}(p, q) = -\tau_{U,1-V}^{LL}(p, 1-q)$. Next,

$$\begin{aligned}\tau_{-X,-Y}^{LL}(F_{-X}^{-1}(1-p), F_{-Y}^{-1}(1-q)) &= \tau_{-X,-Y}^{LL}(-F_X^{-1}(p), -F_Y^{-1}(q)) = \tau_{X,Y}^{UU}(F_X^{-1}(p), F_Y^{-1}(q)) \\ &= \tau_{U,V}^{UU}(p, q).\end{aligned}$$

Since $\tau_{-X,-Y}^{LL}(F_{-X}^{-1}(1-p), F_{-Y}^{-1}(1-q)) = \tau_{1-U,1-V}^{LL}(1-p, 1-q)$, we have

$$\tau_{U,V}^{UU}(p, q) = \tau_{1-U,1-V}^{LL}(1-p, 1-q).$$

Equation $\tau_{U,V}^{UU}(p, q) = \tau_{1-U,1-V}^{LL}(1-p, 1-q)$ has been proved.

Web Appendix B Strategies for Defining Local Regions and a Discussion of Their Advantages and Limitations

Below is a detailed introduction of four strategies for defining local regions:

- (1) Defining regions based on scientific thresholds or quantile intervals. The local regions are determined based on clinically meaningful threshold intervals (e.g., disease stages), quantile intervals, or by directly selecting regions of interest to researchers. For instance, according to current classification criteria (WHO Consultation, 2000; Jones et al., 2025), six stages of body mass index (BMI, kg/m²) are underweight (< 18.50), normal (18.50-24.99), preobese (25.00-29.99), and obese class I, II, III (30.00-34.99, 35.00-39.99, ≥ 40.00), respectively; four stages of systolic blood pressure (SBP, mm Hg) are normal (< 120), elevated (120-129), hypertension stage 1 and 2 (130-139 and ≥ 140), respectively. The region {BMI ∈ [30.00, 34.99], SBP ∈ [130, 139]} can be used to precisely analyze the dependence between BMI and SBP for the {obese class I, hypertension stage I} population. This strategy is recommended when researchers are familiar with the threshold intervals of the scientific variables, or have specific numerical or quantile intervals of interest.
- (2) Constructing a $k_x \times k_y$ grid using clinical stages. If two scientific variables are divided into k_x and k_y stages respectively according to recognized classification criteria, the whole dataset can be partitioned into a grid composed of $k_x \times k_y$ cells, which contains all possible stage combinations. For instance, based on the six stages of BMI and four stages of SBP, a 6×4 grid can be constructed to simultaneously display the local dependence of all subgroups. This strategy is recommended when both variables have recognized numerical stages or grades.

- (3) Partitioning the space into $k \times k$ equal-size cells. The horizontal coordinate axis (of width w_x) and the vertical coordinate axis (of height w_y) are divided into $k \in \mathbb{Z}^+$ equal-length intervals (typically $k = 2-20$) based on raw values or quantiles, respectively, thereby forming a grid composed of $k \times k$ cells. The width and height of each region are $w_x^{cell} = \frac{w_x}{k}$ and $w_y^{cell} = \frac{w_y}{k}$ (or $w_u^{cell} = w_v^{cell} = \frac{1}{k}$ for quantile-based coordinates), respectively. This strategy is suitable for rapid and visual analysis of local dependence patterns across the entire dataset.
- (4) Applying a fixed-size rolling window derived from Strategy 3. An initial region is first determined in the lower-left corner of the coordinate system according to Strategy 3 and then shifted horizontally and vertically by fixed step lengths $\Delta_x = \frac{w_x^{cell}}{m}$ and $\Delta_y = \frac{w_y^{cell}}{m}$ (or $\Delta_u = \Delta_v = \frac{1}{mk}$ for quantile-based coordinates) to the right and upward, respectively, with each shift generating a new region. $m \in \mathbb{Z}^+$ is typically set between 2 and 10 to balance computational precision and traversal efficiency. This operation is repeated for each new region until all $[m(k-1)+1]^2$ regions are traversed. This strategy is suitable for conducting an in-depth analysis of the whole dataset and for identifying target regions (e.g., regions with the strongest or weakest local dependence).

In addition to the four strategies above, researchers can also develop new strategies based on our framework to meet their specific needs.

Web Table 1 further compares the advantages and limitations of above strategies. In practice, the choice of appropriate strategy should balance factors such as interpretability, information retention and computational considerations, guided by the research objectives.

Web Table 1: Advantages and limitations of the four strategies for defining local regions

Strategy	Advantages	Limitations
(1)	<ul style="list-style-type: none"> • Highly flexible, allowing researchers to designate targeted study regions based on variable threshold intervals to align with different scientific questions. • Strongly interpretable. • Convenient for visualization. 	<ul style="list-style-type: none"> • The determination of region width or height varies with the scientific question, leading to inconsistent criteria for defining local regions across different studies. • The selection of regions of interest by researchers is largely dependent on their subjective judgment.
(2)	<ul style="list-style-type: none"> • Easy to implement. • Strongly interpretable. • High computational efficiency. • Convenient for visualization. 	<ul style="list-style-type: none"> • Data points in some regions may be very sparse or even empty, leading to large estimation variance or even making effective estimation impossible.
(3)	<ul style="list-style-type: none"> • Easy to implement. • High computational efficiency. • Convenient for visualization. 	<ul style="list-style-type: none"> • Grid cells are independent and discrete, making it impossible to construct a region centered at arbitrary points. • Incomplete partitioning of local regions makes it impossible to traverse regions between grid cells, leading to the neglect of some local dependence information. • Data points in some regions may be very sparse or even empty, leading to large estimation variance or even making effective estimation impossible.
(4)	<ul style="list-style-type: none"> • Address the issue of missing partial local dependence information in Strategy (3) by traversing more local regions. • Suitable for in-depth and comprehensive analysis of the whole dataset. 	<ul style="list-style-type: none"> • When the width and height of the local region are narrow and the step lengths are small, the number of regions traversed will increase significantly, leading to a decline in traversal efficiency.

Web Appendix C Local Dependence Surfaces for Common Archimedean Copula Models

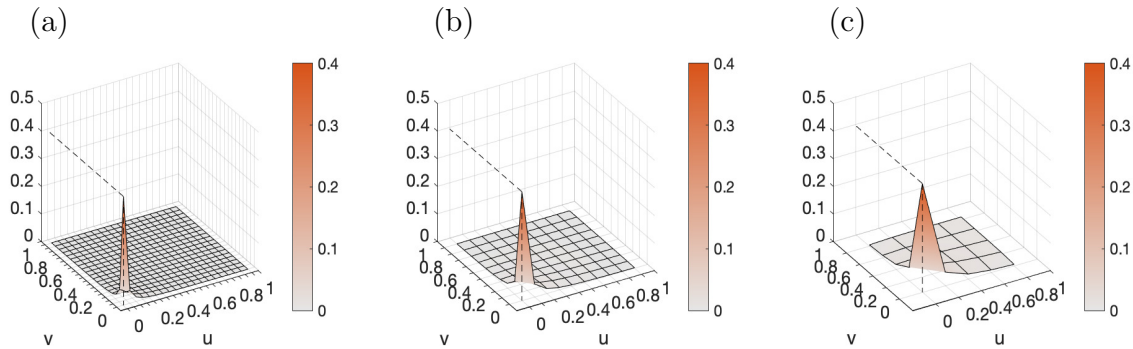
C.1 Type I Local Kendall's τ Surfaces

This subsection plots Type I local dependence surfaces for the Clayton, Gumbel and Frank copula models. In the three-dimensional coordinate system, each point (u, v, z) represents $\tau_{UV}^l(u, u + I, v, v + I) = z$. For instance, when $I = 0.05$, the point $(0, 0, 0.3939)$ represents $\tau_{UV}^l(0, 0.05, 0, 0.05) = 0.3939$. For each copula model, we consider three interval widths $I = 0.05, 0.1$, and 0.2 , respectively.

C.1.1 Type I Local Kendall's τ Surfaces for Clayton Copula Model

Clayton (1978) copula model is a kind of copula with reflection asymmetry and permutation symmetry. It can only capture lower tail dependence, assuming zero upper tail dependence, thus exhibiting an L-shaped dependence structure.

Web Figure 1 presents three Type I local Kendall's τ surfaces of the Clayton copula model, using $C_{Clayton}(u, v; 1.3)$ as an example. It can be seen that local Kendall's τ values in the lower-lower quadrant tail regions (i.e., $\tau_{UV}^l(0, 0.05, 0, 0.05)$, $\tau_{UV}^l(0, 0.1, 0, 0.1)$ and $\tau_{UV}^l(0, 0.2, 0, 0.2)$) are obviously larger than those in other regions. Specifically, local Kendall's τ values in regions near the lower tail always keep at a relatively high level around 0.3939, whereas those in the upper-upper quadrant tail regions (i.e., $\tau_{UV}^l(0.95, 1, 0.95, 1)$, $\tau_{UV}^l(0.9, 1, 0.9, 1)$ and $\tau_{UV}^l(0.8, 1, 0.8, 1)$) are all very small. This indicates that $C_{Clayton}(u, v; 1.3)$ is suitable for characterizing dependence structure with strong local dependence in the lower-lower quadrant tail regions.

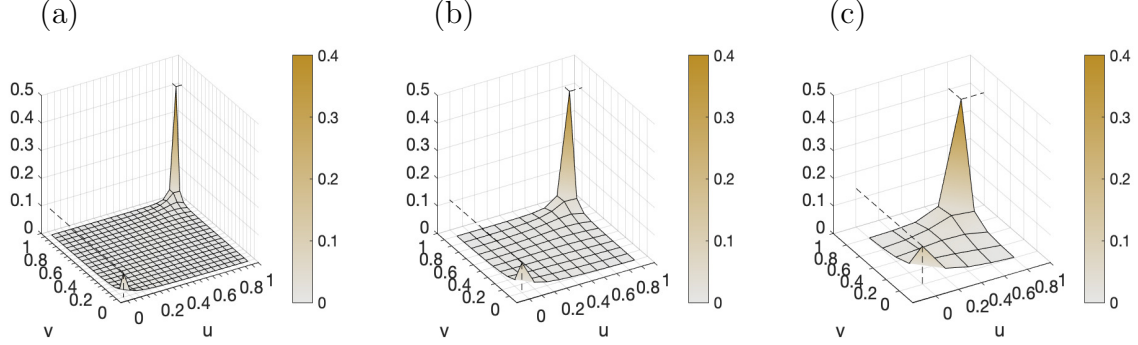


Web Figure 1: Type I local Kendall's τ surfaces of $C_{Clayton}(u, v; 1.3)$. (a) Left plot: $I = 0.05$; (b) Middle plot: $I = 0.1$; (c) Right plot: $I = 0.2$.

C.1.2 Type I Local Kendall's τ Surfaces for Gumbel Copula Model

Gumbel (1960) copula is also a copula model with reflection asymmetry and permutation symmetry. However, it captures only the upper tail dependence and assumes that the lower tail dependence is zero, thus exhibiting a J-shaped dependence structure.

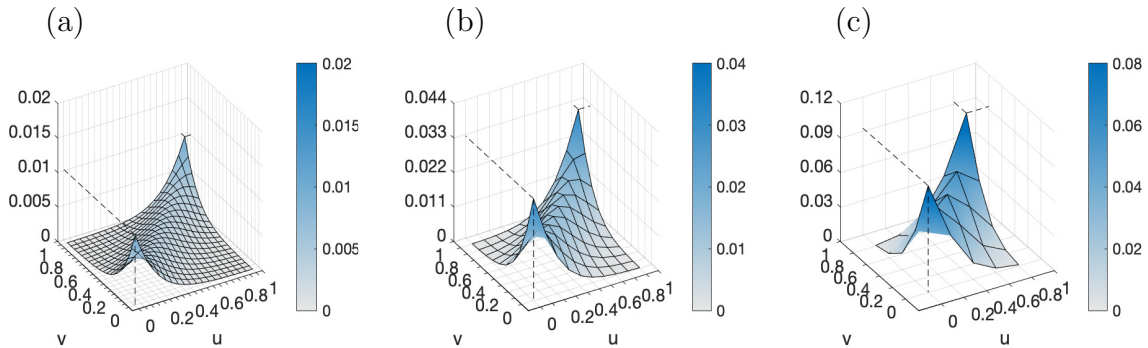
Web Figure 2 presents three Type I local Kendall's τ surfaces of the Gumbel copula model, using $C_{Gumbel}(u, v; 2)$ as an example. Unlike $C_{Clayton}(u, v; 1.3)$, local Kendall's τ values of $C_{Gumbel}(u, v; 2)$ in the upper-upper quadrant tail regions (i.e., $[0.95, 1] \times [0.95, 1]$, $[0.9, 1] \times [0.9, 1]$, and $[0.8, 1] \times [0.8, 1]$) are obviously larger than those in other regions, indicating that $C_{Gumbel}(u, v; 2)$ is appropriate for characterizing dependence structure with strong local dependence in the upper-upper quadrant tail regions.



Web Figure 2: Type I local Kendall's τ surfaces of $C_{Gumbel}(u, v; 2)$. (a) Left plot: $I = 0.05$; (b) Middle plot: $I = 0.1$; (c) Right plot: $I = 0.2$.

C.1.3 Type I Local Kendall's τ Surfaces for Frank Copula Model

Frank (1979) copula is symmetric along both the main and minor diagonals but has no extreme-value dependence. Web Figure 3 presents three Type I local Kendall's τ surfaces of the Frank copula model, using $C_{Frank}(u, v; 7)$ as an example. Unlike $C_{Clayton}(u, v; 1.3)$ and $C_{Gumbel}(u, v; 2)$, its local Kendall's τ surfaces exhibit symmetric characteristics both along the main and minor diagonals. Local Kendall's τ values in the two regions at the ends of the main diagonal are obviously larger than those in other regions, indicating that $C_{Frank}(u, v; 7)$ is suitable for modeling dependence structure with symmetric local dependence in the upper-upper and lower-lower quadrant tail regions.



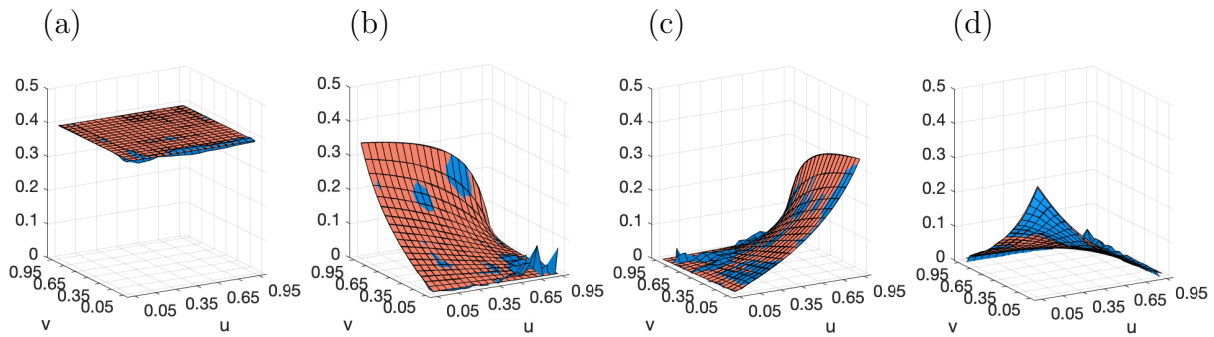
Web Figure 3: Type I local Kendall's τ surfaces of $C_{Frank}(u, v; 7)$. (a) Left plot: $I = 0.05$; (b) Middle plot: $I = 0.1$; (c) Right plot: $I = 0.2$.

C.2 Type II Local Kendall's τ Surfaces

C.2.1 Type II Local Kendall's τ Surfaces for Clayton Copula Model

Web Figures 4(a)-(d) illustrate four Type II local Kendall's τ surfaces of $C_{Clayton}(u, v; 1.3)$. It can be seen that all four theoretical local Kendall's τ surfaces are highly consistent with their corresponding empirical surfaces. Web Figures 4(a) and (d) indicate that the theoretical lower-lower local Kendall's τ surface always keeps at a relatively high level (around 0.3939), while the theoretical upper-upper local Kendall's τ surface exhibits obvious decreasing trends as $u \rightarrow 1$ and $v \rightarrow 1$, and gradually tends to 0. This implies that the lower-lower Kendall's τ of $C_{Clayton}(u, v; 1.3)$ is obviously stronger, which is also proved by its Type I local Kendall's τ surfaces presented in subsection C.1.1. In addition, both surfaces are symmetric along the main diagonal (i.e., from point (0,0,0) to (1,1,0)). Web Figures 4(b) and (c) show that the theoretical upper-lower and lower-upper local Kendall's τ surfaces of $C_{Clayton}(u, v; 1.3)$ are very similar, and both exhibit obvious decreasing trends as $u \rightarrow 1, v \rightarrow 0$ and $u \rightarrow 0, v \rightarrow 1$, respectively. Unlike the upper-upper and lower-lower local Kendall's τ surfaces, however, both surfaces are asymmetric along the minor diagonal (i.e., from point (0,1,0) to (1,0,0)).

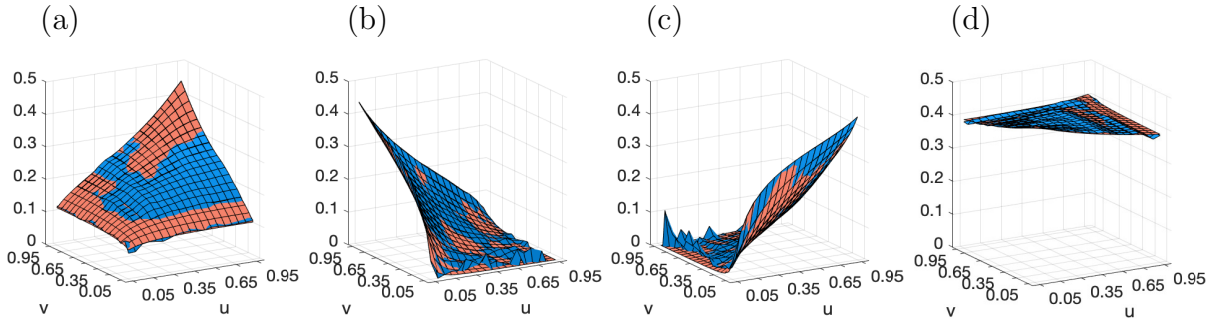
We calculate the global Kendall's τ of $C_{Clayton}(u, v; 1.3)$ using formula $\tau = \theta_C / (\theta_C + 2)$ (Nelsen, 2006, p. 163), which yields $\tau \approx 0.3939$. As can be seen from Web Figures 4(b)(c) and (d), the global Kendall's τ is obviously larger than the upper-upper, upper-lower, and lower-upper local Kendall's τ . We also note that as $u \rightarrow 1, v \rightarrow 0$ and $u \rightarrow 0, v \rightarrow 1$, both the upper-lower and lower-upper empirical local Kendall's τ surfaces fluctuate more dramatically. This is primarily due to the relatively small sample size in these regions. Increasing the simulation sample size to enhance data density would effectively reduce these fluctuations.



Web Figure 4: Type II local Kendall's τ surfaces of $C_{Clayton}(u, v; 1.3)$. (a) Lower-lower local Kendall's τ surfaces. (b) Upper-lower local Kendall's τ surfaces. (c) Lower-upper local Kendall's τ surfaces. (d) Upper-upper local Kendall's τ surfaces. The smooth (red) and non-smooth (blue) curved surfaces represent theoretical and empirical local Kendall's τ surfaces, respectively.

C.2.2 Type II Local Kendall's τ Surfaces for Gumbel Copula Model

Web Figures 5(a)-(d) show four Type II local Kendall's τ surfaces of $C_{Gumbel}(u, v; 2)$. As can be seen, all four theoretical local Kendall's τ surfaces also fit very well with their corresponding empirical surfaces. Web Figures 5(a) and (d) indicate that both theoretical lower-lower and upper-upper local Kendall's τ surfaces are symmetric along the main diagonal. However, unlike $C_{Clayton}(u, v; 1.3)$, the upper-upper local Kendall's τ surface of $C_{Gumbel}(u, v; 2)$ is overall higher than its lower-lower local Kendall's τ surface. Specifically, the theoretical upper-upper local Kendall's τ surface remains at a relatively high level that is between 0.38 and 0.48, whereas the theoretical lower-lower local Kendall's τ surface decreases from around 0.45 to around 0.09 gradually as $u, v \rightarrow 0.05$. Web Figures 5(b) and (c) indicate that both the upper-lower and lower-upper local Kendall's surfaces show obvious decreasing trends. Like $C_{Clayton}(u, v; 1.3)$, neither of the two surfaces is symmetric along the minor diagonal.

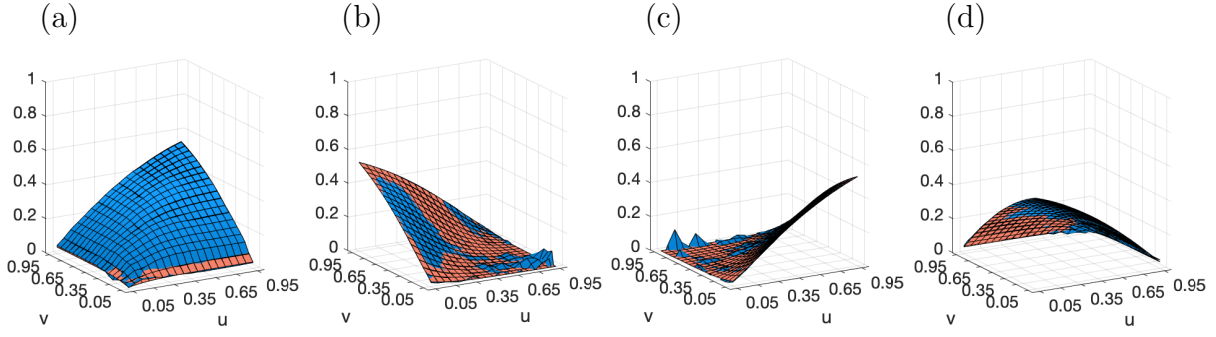


Web Figure 5: Type II local Kendall's τ surfaces of $C_{Gumbel}(u, v; 2)$. (a) Lower-lower local Kendall's τ surfaces. (b) Upper-lower local Kendall's τ surfaces. (c) Lower-upper local Kendall's τ surfaces. (d) Upper-upper local Kendall's τ surfaces. The smooth (red) and non-smooth (blue) curved surfaces represent theoretical and empirical local Kendall's τ surfaces, respectively.

We calculate the global Kendall's τ of $C_{Gumbel}(u, v; 2)$ using formula $\tau = 1 - 1/\theta_G$ (Genest and Rivest, 1993), which gives $\tau = 0.5$. Web Figures 5(a)(b)(c) demonstrate that the global Kendall's τ is obviously larger than the lower-lower, upper-lower and lower-upper local Kendall's τ .

C.2.3 Type II Local Kendall's τ Surfaces for Frank Copula Model

Web Figures 6(a)-(d) display four Type II local Kendall's τ surfaces of $C_{Frank}(u, v; 7)$. All four surfaces exhibit clear decreasing trends along the diagonals, eventually approaching zero. In addition, both the theoretical upper-upper and lower-lower local Kendall's τ surfaces are symmetric along the main diagonal, while both the theoretical upper-lower and lower-upper local Kendall's τ surfaces are symmetric along the minor diagonal.



Web Figure 6: Type II local Kendall's τ surfaces of $C_{Frank}(u, v; 7)$. (a) Lower-lower local Kendall's τ surfaces. (b) Upper-lower local Kendall's τ surfaces. (c) Lower-upper local Kendall's τ surfaces. (d) Upper-upper local Kendall's τ surfaces. The smooth (red) and non-smooth (blue) curved surfaces represent theoretical and empirical local Kendall's τ surfaces, respectively.

Web Appendix D Comparative Analysis of Global and Local Kendall's τ for Different Copula Models

This section presents the comparison results of global and generalized local Kendall's τ for the Gumbel, Clayton, Frank, and FGM copula models.

D.1 Comparison Results for Gumbel Copula Model

Web Figure 7 compares τ_{global} and generalized local Kendall's τ (with different square region widths) for Gumbel copula models with parameters $\theta_G = 1.1, 1.22, 1.5$, and 3 , respectively. Web Figures 7(a)(b)(e)(f)(i)(j)(m) and (n) show that when $\theta_G = 1.1$ and 1.22 and $I = 0.05, 0.1, 0.2$, and 0.25 , $\tau_{UV}^{UU}(0.95, 0.95)$, $\tau_{UV}^{UU}(0.9, 0.9)$, $\tau_{UV}^{UU}(0.8, 0.8)$, and $\tau_{UV}^{UU}(0.75, 0.75)$ are all significantly larger than τ_{global} . Web Figure 7(q) indicates that when $\theta_G = 1.1$ and $I = 0.5$, $\tau_{UV}^{UU}(0.5, 0.5)$ is very close to τ_{global} . In contrast, Web Figure 7(c)(d)(g)(h)(k)(l)(o)(p)(s) and (t) demonstrate that when $\theta_G = 1.5$ and 3 , local τ in all regions is smaller than τ_{global} . These findings suggest that $\tau_{XY,q}^{UU}$ of the Gumbel copula model may be greater than, equal to, or less than its τ_{global} , depending on the copula parameter θ_G and the region width I .

We further compare τ_{global} with two types of quantile Kendall's τ — $\tau_{XY,q}^{UU}$ and $\tau_{XY,q}^{LL}$ —and obtain consistent findings. Web Figures 8(a)-(f) present the τ_{global} and upper-upper quantile Kendall's τ curves for Gumbel copula model with parameters $\theta_G = 1.1$ and $1.22, 1.5, 3, 5$, and 10 , respectively. It is observed that when $\theta_G = 1.5, 3, 5$, and 10 , all $\tau_{XY,q}^{UU}$ values are smaller than τ_{global} . As the quantile $q \rightarrow 0$, $\tau_{XY,q}^{UU}$ gradually increases to τ_{global} . However, when $\theta_G = 1.1$ and 1.22 (i.e., when the overall dependence is weaker), $\tau_{XY,q}^{UU}$ with larger q is greater than τ_{global} . As q decreases, $\tau_{XY,q}^{UU}$ first gradually drops below τ_{global} , and then increases to τ_{global} again. This suggests that, for the Gumbel copula model, $\tau_{XY,q}^{UU}$ may be greater than, equal to, or less than τ_{global} . Conversely, Web Figures 8(g)-(l) indicate that $\tau_{XY,q}^{LL}$ of the Gumbel copula model with different parameters remains consistently smaller

than τ_{global} . In addition, we also observe that the two types of empirical quantile Kendall's τ calculated using simulated data highly coincide with those estimated by the copula model, further verifying the correctness of the aforementioned conclusions.

D.2 Comparison Results for Clayton Copula Model

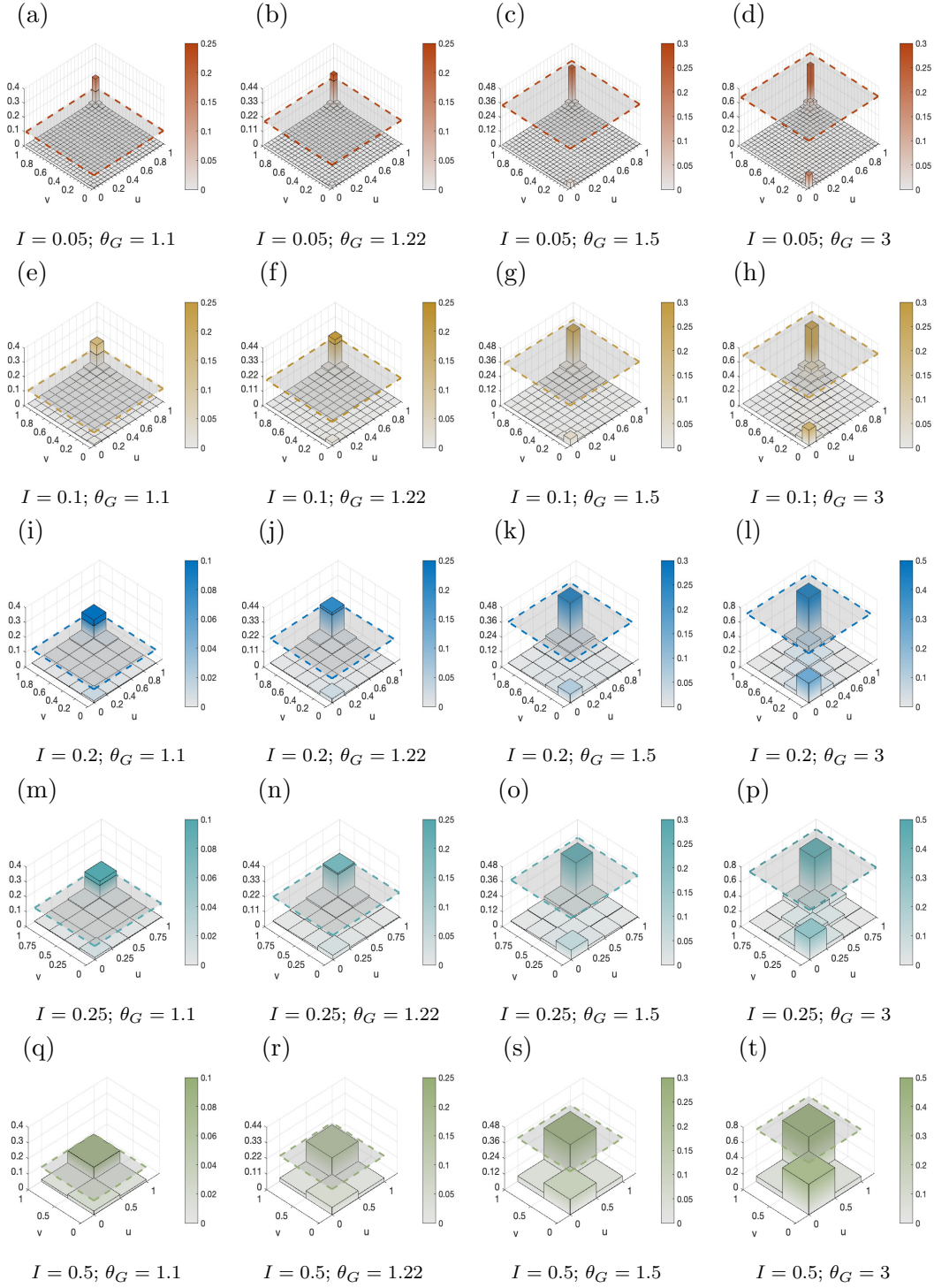
Web Figure 9 compares τ_{global} and generalized local Kendall's τ (with different region widths) for Clayton copula models with parameters $\theta_C = 0.1, 0.2, 0.5$, and 2 , respectively. It can be observed that, for all Clayton copula models, $\tau_{UV}^{LL}(0.05, 0.05)$, $\tau_{UV}^{LL}(0.1, 0.1)$, $\tau_{UV}^{LL}(0.2, 0.2)$, $\tau_{UV}^{LL}(0.25, 0.25)$, and $\tau_{UV}^{LL}(0.5, 0.5)$ are all equal to τ_{global} . In contrast, local Kendall's τ values in other regions are obviously smaller than τ_{global} . Web Figures 10(a)-(f) and 10(g)-(l) further confirm that, $\tau_{XY,q}^{UU}$ of the Clayton copula model is smaller than its τ_{global} , while $\tau_{XY,q}^{LL}$ equals τ_{global} .

D.3 Comparison Results for Frank Copula Model

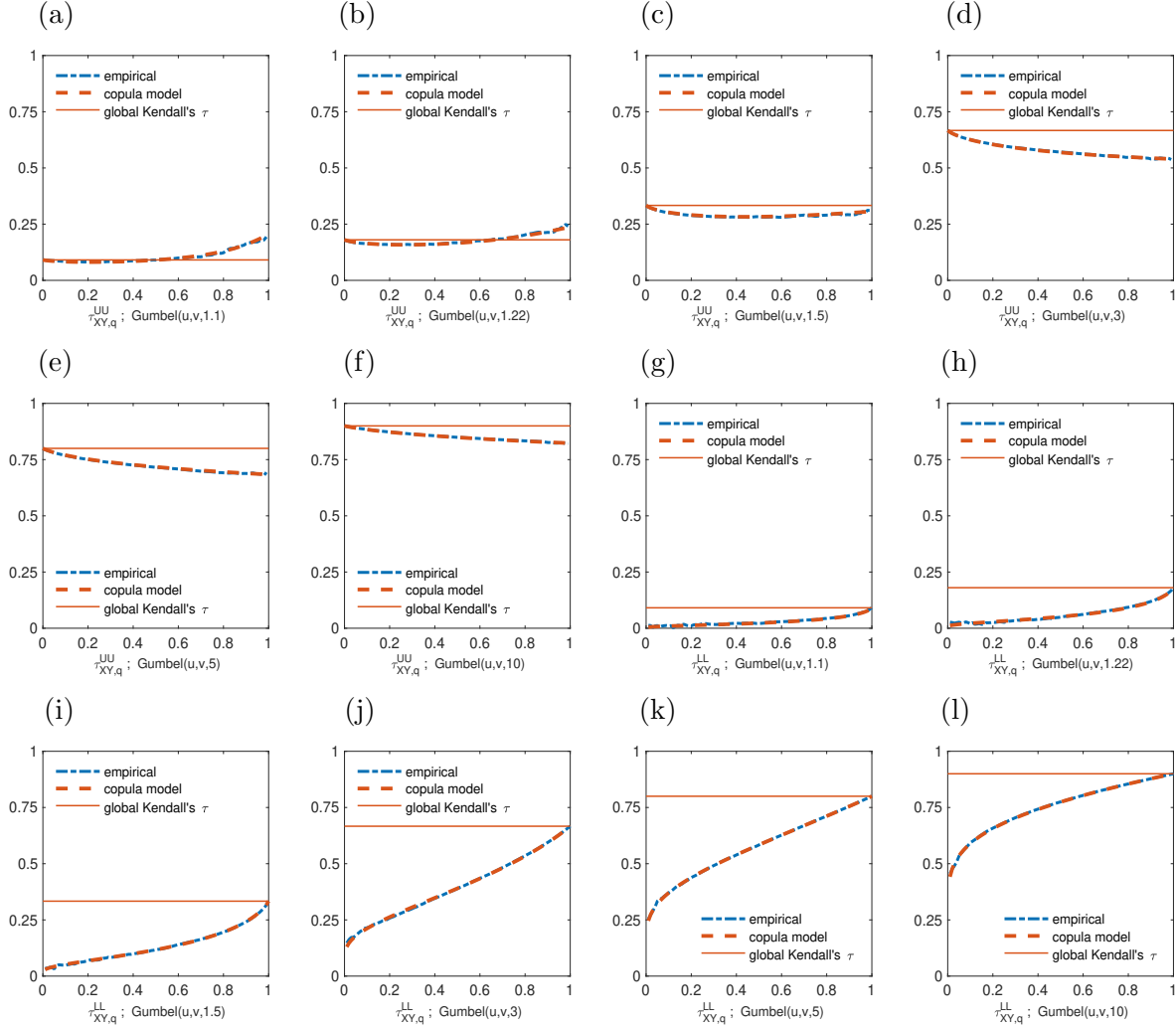
Web Figure 11 compares τ_{global} and generalized local Kendall's τ (with different region widths) for Frank copula models with parameters $\theta_F = 0.3, 3, 8$, and 15 , respectively. It is observed that local Kendall's τ in all regions is markedly smaller than τ_{global} . Web Figures 12(a)-(f) and 12(g)-(l) further indicate that both $\tau_{XY,q}^{UU}$ and $\tau_{XY,q}^{LL}$ of the Frank copula model are smaller than its τ_{global} .

D.4 Comparison Results for FGM Copula Model

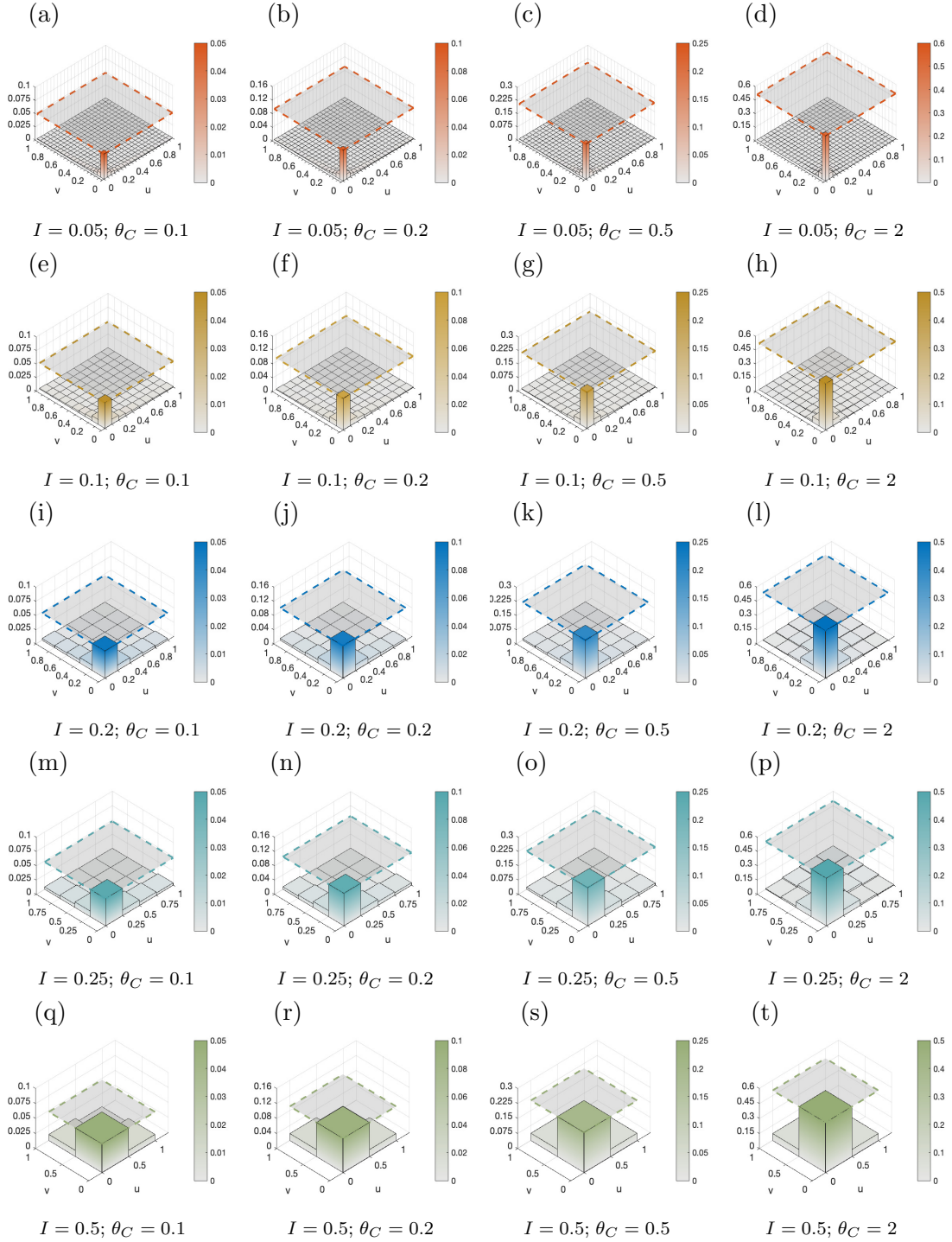
Web Figure 13 compares τ_{global} and generalized local Kendall's τ (with different region widths) for FGM copula models with parameters $\alpha = 0.1, 0.3, 0.5$, and 0.7 , respectively. It can be seen that local Kendall's τ in all regions is noticeably smaller than τ_{global} . Web Figures 14(a)-(f) and 14(g)-(l) further indicate that both $\tau_{XY,q}^{UU}$ and $\tau_{XY,q}^{LL}$ of the FGM copula model are smaller than its τ_{global} .



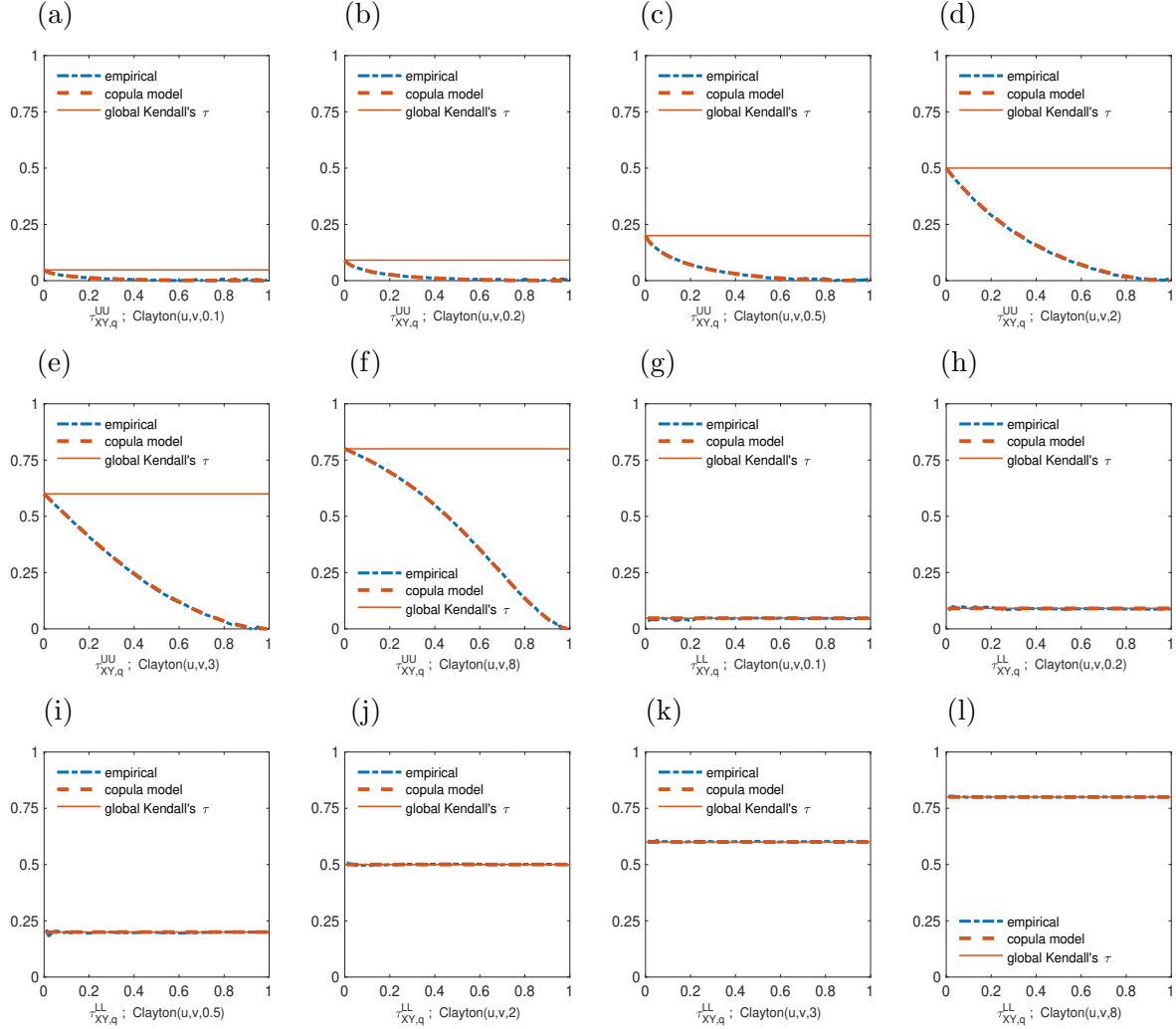
Web Figure 7: Comparison of τ_{global} and generalized local Kendall's τ (with different square region widths) for Gumbel copula models under different association parameters. (a)-(d): $I = 0.05$, $\theta_G = 1.1, 1.22, 1.5$, and 3 ; (e)-(h): $I = 0.1$, $\theta_G = 1.1, 1.22, 1.5$, and 3 ; (i)-(l): $I = 0.2$, $\theta_G = 1.1, 1.22, 1.5$, and 3 ; (m)-(p): $I = 0.25$, $\theta_G = 1.1, 1.22, 1.5$, and 3 ; (q)-(t): $I = 0.5$, $\theta_G = 1.1, 1.22, 1.5$, and 3 . The region enclosed by the dashed lines represents the global Kendall's τ plane.



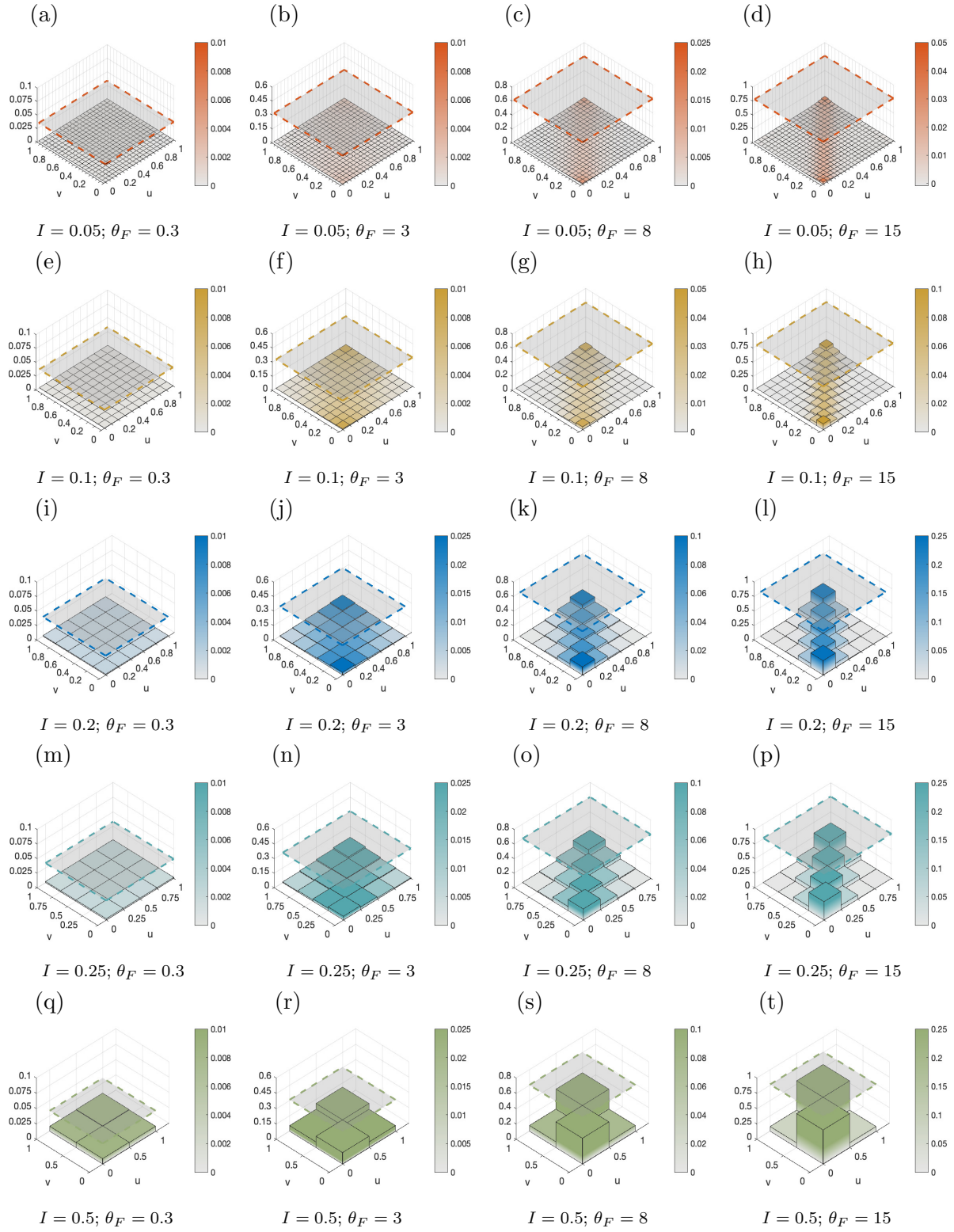
Web Figure 8: Comparison of τ_{global} and quantile Kendall's τ for Gumbel copula models under different association parameters. The horizontal axis: quantile q . (a)-(f): upper-upper quantile Kendall's τ , $q \in [0, 0.99]$; $\theta_G = 1.1, 1.22, 1.5, 3, 5$, and 10 , respectively. (g)-(l): lower-lower quantile Kendall's τ , $q \in [0.01, 1]$; $\theta_G = 1.1, 1.22, 1.5, 3, 5$, and 10 , respectively. In each figure, the red solid line represents global Kendall's τ estimated by the copula model. The red dash line represents quantile Kendall's τ estimated by the copula model. The blue dash-dot line represents empirical quantile Kendall's τ estimated by 300,000 random samples generated by the copula model.



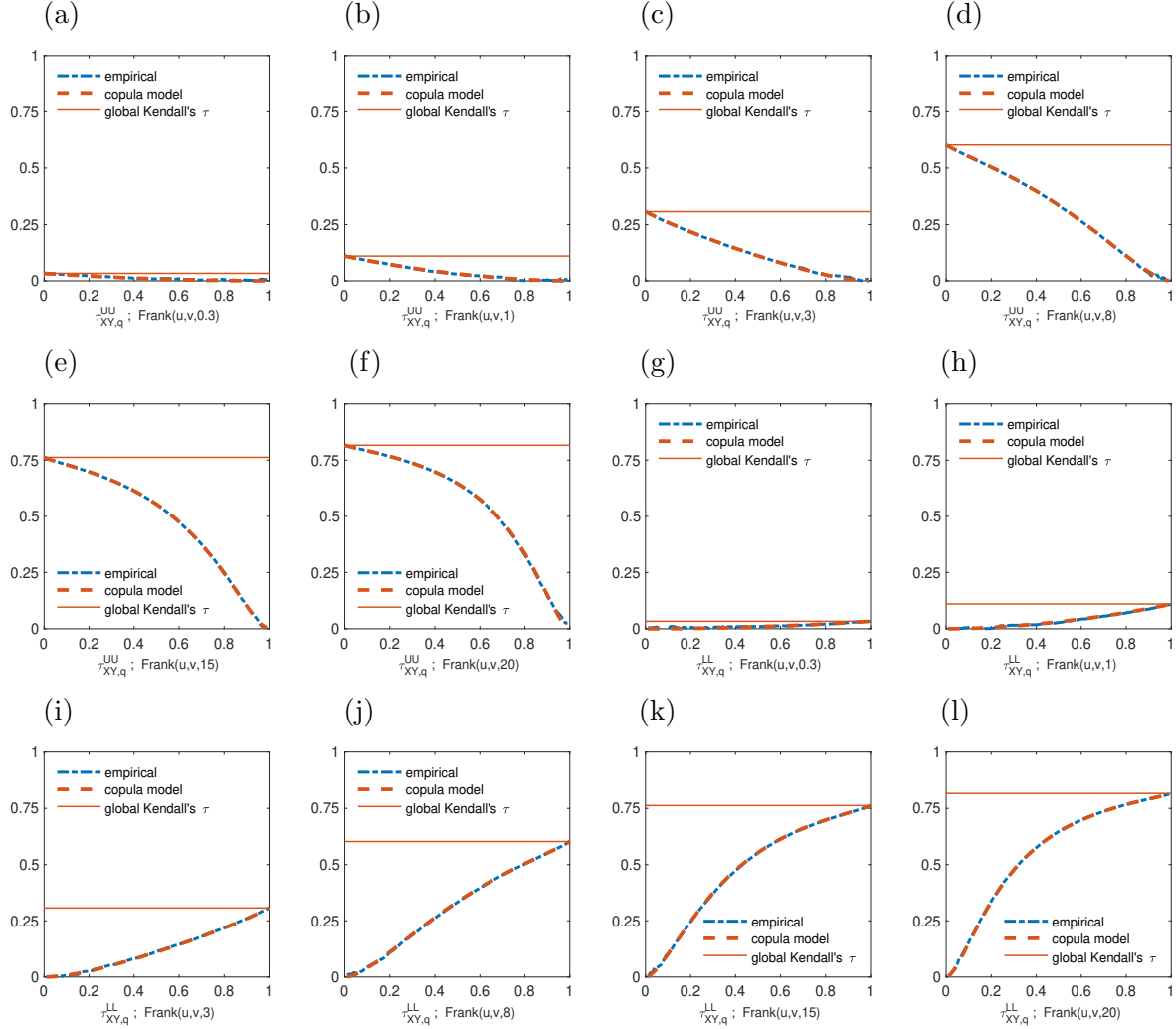
Web Figure 9: Comparison of τ_{global} and generalized local Kendall's τ (with different region widths) for Clayton copula models under different association parameters. (a)-(d): $I = 0.05$, $\theta_C = 0.1, 0.2, 0.5$, and 2 ; (e)-(h): $I = 0.1$, $\theta_C = 0.1, 0.2, 0.5$, and 2 ; (i)-(l): $I = 0.2$, $\theta_C = 0.1, 0.2, 0.5$, and 2 ; (m)-(p): $I = 0.25$, $\theta_C = 0.1, 0.2, 0.5$, and 2 ; (q)-(t): $I = 0.5$, $\theta_C = 0.1, 0.2, 0.5$, and 2 . The region enclosed by the dashed lines represents the global Kendall's τ plane.



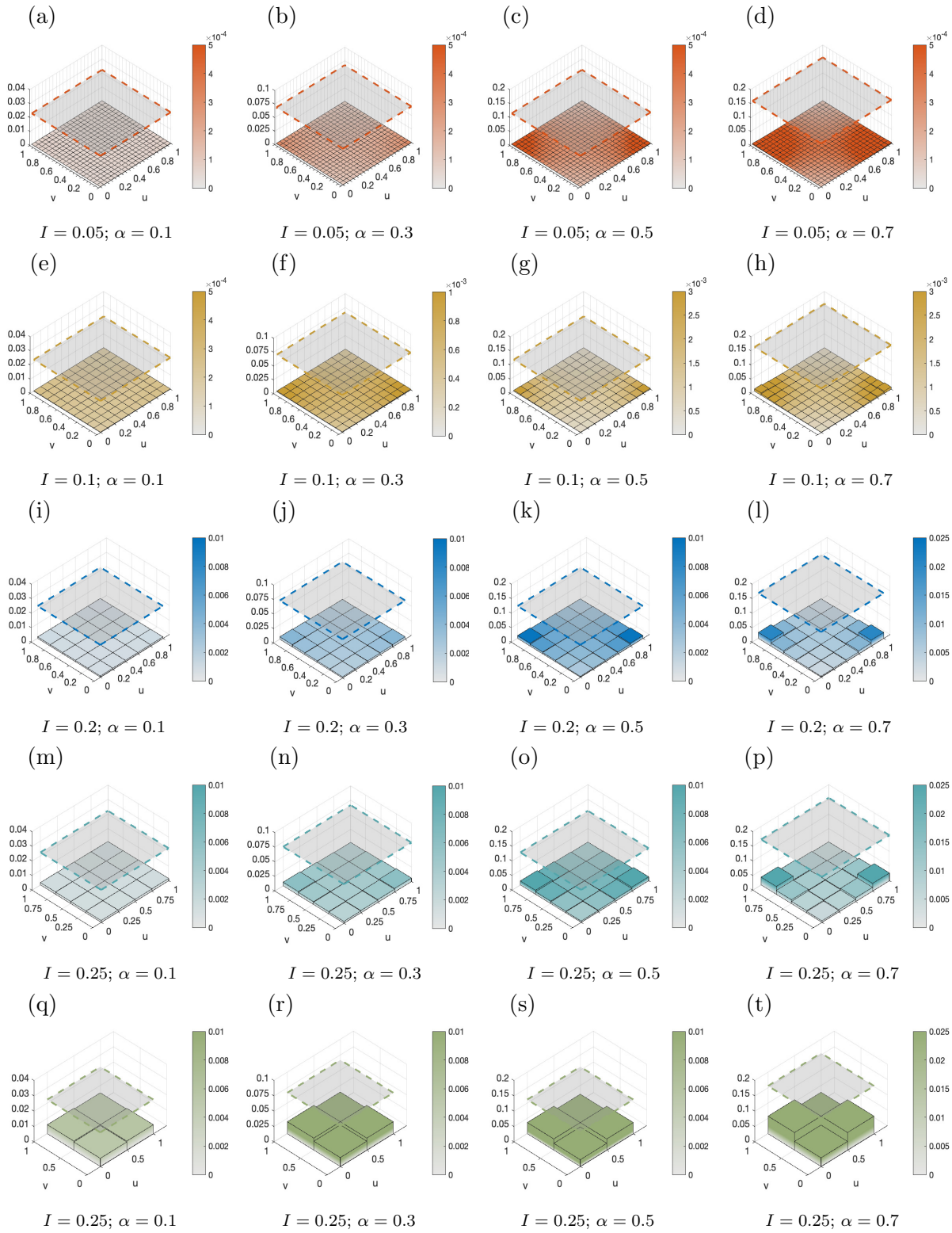
Web Figure 10: Comparison of τ_{global} and quantile Kendall's τ for Clayton copula models with different association parameters. The horizontal axis: quantile q . (a)-(f): upper-upper quantile Kendall's τ , $q \in [0, 0.99]$; $\theta_C = 0.1, 0.2, 0.5, 2, 3$, and 8 , respectively. (g)-(l): lower-lower quantile Kendall's τ , $q \in [0.01, 1]$; $\theta_C = 0.1, 0.2, 0.5, 2, 3$, and 8 , respectively. The red solid lines represent global Kendall's τ estimated by the copula model. The red dash lines represent quantile Kendall's τ estimated by the copula model. The blue dash-dot lines represent empirical quantile Kendall's τ estimated by 300,000 random samples generated by the copula model.



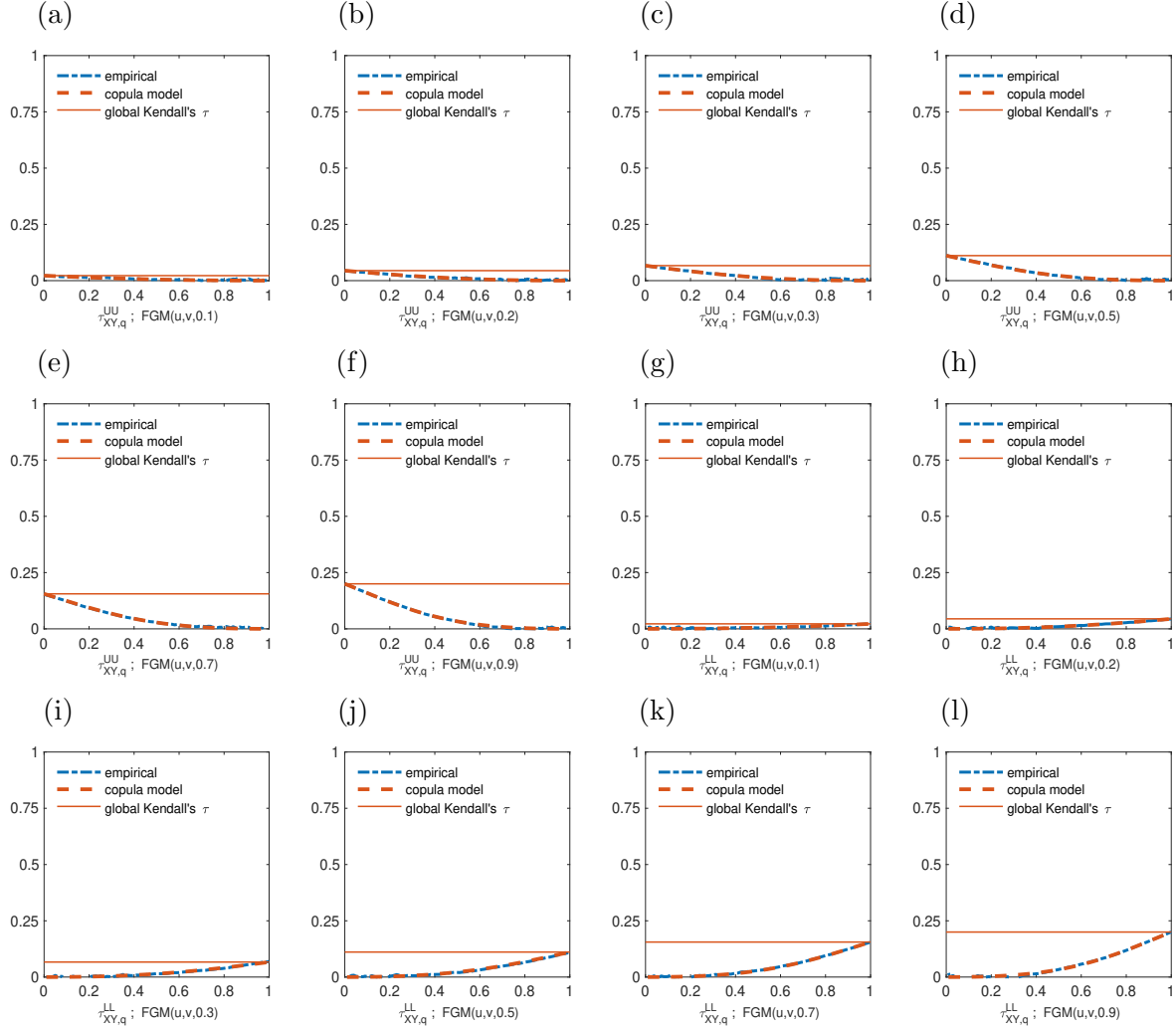
Web Figure 11: Comparison of τ_{global} and generalized local Kendall's τ (with different region widths) for Frank copula models under different association parameters. (a)-(d): $I = 0.05$, Frank copula parameter $\theta_F = 0.3, 3, 8$, and 15 ; (e)-(h): $I = 0.1$, $\theta_F = 0.3, 3, 8$, and 15 ; (i)-(l): $I = 0.2$, $\theta_F = 0.3, 3, 8$, and 15 ; (m)-(p): $I = 0.25$, $\theta_F = 0.3, 3, 8$, and 15 ; (q)-(t): $I = 0.5$, $\theta_F = 0.3, 3, 8$, and 15 . The region enclosed by the dashed lines represents the global Kendall's τ plane.



Web Figure 12: Comparison of τ_{global} and quantile Kendall's τ for Frank copula models with different association parameters. The horizontal axis: quantile q . (a)-(f): upper-upper quantile Kendall's τ , $q \in [0, 0.99]$; $\theta_F = 0.3, 1, 3, 8, 15$, and 20 , respectively. (g)-(l): lower-lower quantile Kendall's τ , $q \in [0.01, 1]$; $\theta_F = 0.3, 1, 3, 8, 15$, and 20 , respectively. The red solid lines represent global Kendall's τ estimated by the copula model. The red dash lines represent quantile Kendall's τ estimated by the copula model. The blue dash-dot lines represent empirical quantile Kendall's τ estimated by 300,000 random samples generated by the copula model.



Web Figure 13: Comparison of τ_{global} and generalized local Kendall's τ (with different region widths) for FGM copula models under different association parameters. (a)-(d): $I = 0.05$, $\alpha = 0.1, 0.3, 0.5$, and 0.7 ; (e)-(h): $I = 0.1$, $\alpha = 0.1, 0.3, 0.5$, and 0.7 ; (i)-(l): $I = 0.2$, $\alpha = 0.1, 0.3, 0.5$, and 0.7 ; (m)-(p): $I = 0.25$, $\alpha = 0.1, 0.3, 0.5$, and 0.7 ; (q)-(t): $I = 0.5$, $\alpha = 0.1, 0.3, 0.5$, and 0.7 . The region enclosed by the dashed lines represents the global Kendall's τ plane.



Web Figure 14: Comparison of τ_{global} and quantile Kendall's τ for FGM copula models with different association parameters. The horizontal axis: quantile q . (a)-(f): upper-upper quantile Kendall's τ , $q \in [0, 0.99]$; $\alpha = 0.1, 0.2, 0.3, 0.5, 0.7$, and 0.9 , respectively. (g)-(l): lower-lower quantile Kendall's τ , $q \in [0.01, 1]$; $\alpha = 0.1, 0.2, 0.3, 0.5, 0.7$, and 0.9 , respectively. The red solid lines represent global Kendall's τ estimated by the copula model. The red dash lines represent quantile Kendall's τ estimated by the copula model. The blue dash-dot lines represent empirical quantile Kendall's τ estimated by 300,000 random samples generated by the copula model.

Web Appendix E Additional Simulation Results

E.1 Performance Metrics of Two Estimators

Web Tables 2-5 report the empirical biases (Bias), empirical standard errors (SE_e), average bootstrap standard errors (ASE_b) and empirical 95% coverage probabilities (CP%) of the two estimators in different regions for sample sizes $n = 100, 200, 1000$, and 2000 , respectively. Web Figure 15 further graphically compares their empirical standard errors and 95% coverage probabilities for $n = 100, 200, 500, 1000$, and 2000 , respectively.

Web Table 2: Comparison of performance metrics for the two estimators under different copula models with sample size $n = 100$

Region	Model	Parameter	U-statistic-based estimator				Copula model-based estimator			
			Bias	SE_e	ASE_b	CP%	Bias	SE_e	ASE_b	CP%
I. $(-\infty, -2] \times (-\infty, -2]$	Clayton	1	0.006	0.241	0.210	85.5	0.034	0.037	0.033	77.7
		2	0.017	0.191	0.180	88.2	0.012	0.032	0.030	89.0
		4	-0.004	0.159	0.153	90.5	0.004	0.024	0.023	92.6
II. $(-\infty, -1] \times (-\infty, -1]$	Clayton	1	0.003	0.208	0.200	91.5	0.009	0.040	0.039	92.3
		2	<0.001	0.164	0.170	92.7	<0.001	0.034	0.033	93.7
		4	-0.002	0.124	0.133	93.8	-0.001	0.025	0.024	93.4
	Gumbel	1.3	-0.003	0.238	0.218	89.1	0.010	0.014	0.014	92.7
		2	-0.002	0.222	0.212	90.4	0.006	0.021	0.021	94.6
		3	-0.008	0.206	0.200	91.3	0.003	0.026	0.027	95.2
	Frank	1	-0.006	0.238	0.219	88.8	0.005	0.004	0.004	94.0
		4	-0.004	0.236	0.215	88.5	0.004	0.009	0.009	94.7
		7	0.007	0.217	0.209	89.9	0.004	0.014	0.014	94.8
III. $[2, +\infty) \times [2, +\infty)$	Gumbel	1.3	0.010	0.251	0.213	86.4	0.036	0.023	0.022	65.3
		2	0.005	0.228	0.205	86.8	0.014	0.027	0.026	90.1
		3	-0.009	0.195	0.179	90.2	0.007	0.026	0.025	92.0
IV. $[1, +\infty) \times [1, +\infty)$	Clayton	1	0.015	0.241	0.220	88.0	<0.001	0.002	0.002	95.6
		2	0.001	0.239	0.215	88.8	0.001	0.003	0.004	96.6
		4	-0.003	0.229	0.212	89.9	0.002	0.008	0.007	96.0
	Gumbel	1.3	<0.001	0.221	0.210	90.1	0.017	0.029	0.030	91.6
		2	-0.007	0.199	0.187	90.4	0.003	0.029	0.029	94.9
		3	<0.001	0.157	0.159	93.8	0.001	0.027	0.026	93.5
	Frank	1	0.001	0.248	0.217	88.0	0.004	0.004	0.005	93.8
		4	0.004	0.236	0.215	89.0	0.005	0.009	0.009	95.1
		7	0.006	0.216	0.209	90.7	0.003	0.014	0.014	95.4
	Clayton	1	-0.004	0.104	0.104	94.0	0.003	0.027	0.026	93.5
		2	<0.001	0.092	0.094	94.0	0.001	0.035	0.034	93.9
		4	-0.003	0.073	0.075	95.0	<0.001	0.037	0.035	94.0
V. $[-1, 1] \times [-1, 1]$	Gumbel	1.3	0.002	0.106	0.107	94.1	0.001	0.024	0.024	94.3
		2	0.006	0.091	0.095	94.1	0.003	0.034	0.034	94.4
		3	-0.002	0.075	0.079	94.9	<0.001	0.036	0.036	94.7
	Frank	1	-0.002	0.111	0.111	93.7	0.002	0.028	0.026	90.9
		4	0.004	0.104	0.101	93.4	0.003	0.034	0.035	95.9
		7	0.003	0.085	0.087	95.6	0.004	0.038	0.039	94.9

Web Table 3: Comparison of performance metrics for the two estimators under different copula models with sample size $n = 200$

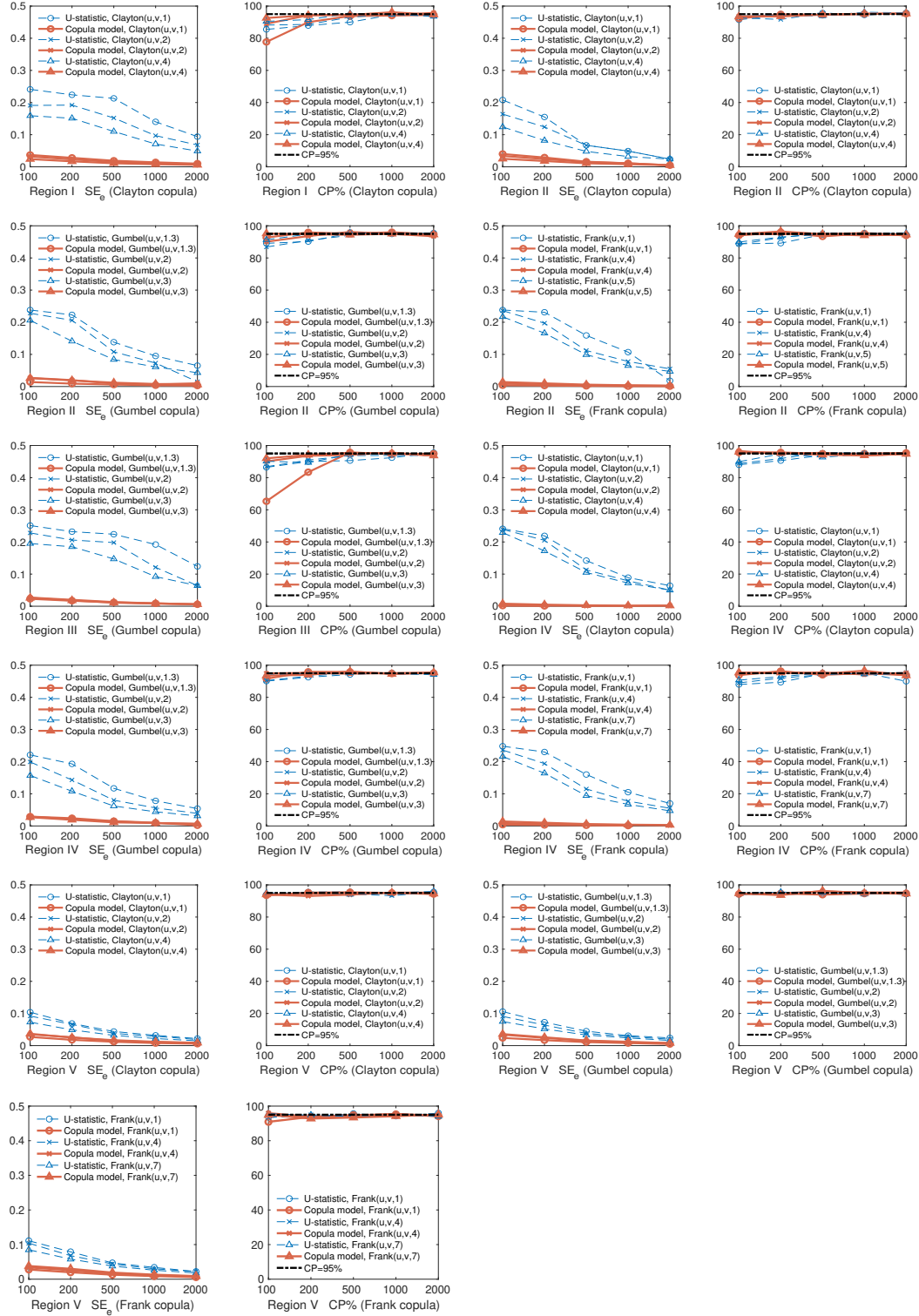
Region	Model	Parameter	U-statistic-based estimator				Copula model-based estimator			
			Bias	SE _e	ASE _b	CP%	Bias	SE _e	ASE _b	CP%
I. $(-\infty, -2] \times (-\infty, -2]$	Clayton	1	-0.010	0.225	0.205	87.9	0.014	0.028	0.027	90.0
		2	0.021	0.192	0.177	88.6	0.004	0.023	0.023	94.3
		4	<0.001	0.151	0.150	91.6	0.001	0.017	0.017	94.3
II. $(-\infty, -1] \times (-\infty, -1]$	Clayton	1	-0.001	0.155	0.160	94.3	0.002	0.029	0.030	94.9
		2	0.001	0.124	0.121	91.7	0.003	0.024	0.024	93.9
		4	0.004	0.082	0.087	94.3	<0.001	0.018	0.017	93.6
	Gumbel	1.3	0.005	0.223	0.209	90.2	0.002	0.009	0.009	95.9
		2	0.004	0.176	0.182	93.2	<0.001	0.015	0.015	94.9
		3	0.002	0.141	0.152	94.8	<0.001	0.019	0.019	94.7
	Frank	1	-0.006	0.231	0.213	89.2	0.001	0.003	0.003	96.0
		4	-0.003	0.197	0.192	92.6	<0.001	0.006	0.006	94.8
		7	0.004	0.166	0.168	93.0	<0.001	0.010	0.010	96.5
III. $[2, +\infty) \times [2, +\infty)$	Gumbel	1.3	-0.005	0.232	0.214	90.0	0.018	0.017	0.017	83.4
		2	<0.001	0.206	0.197	90.9	0.005	0.020	0.020	93.6
		3	0.003	0.185	0.173	89.4	0.002	0.018	0.018	94.2
IV. $[1, +\infty) \times [1, +\infty)$	Clayton	1	<0.001	0.219	0.209	90.6	<0.001	0.001	0.001	95.7
		2	0.004	0.205	0.196	91.9	<0.001	0.002	0.002	94.7
		4	0.002	0.172	0.177	94.8	<0.001	0.005	0.005	95.3
	Gumbel	1.3	<0.001	0.193	0.192	92.5	<0.001	0.024	0.024	95.9
		2	<0.001	0.143	0.142	93.2	0.001	0.021	0.021	95.2
		3	-0.002	0.108	0.111	94.9	<0.001	0.019	0.018	94.0
	Frank	1	0.002	0.230	0.213	89.4	0.001	0.003	0.003	96.3
		4	0.003	0.194	0.191	92.0	<0.001	0.006	0.006	95.2
		7	-0.001	0.164	0.168	92.9	0.001	0.010	0.010	95.4
	Clayton	1	-0.002	0.069	0.071	95.6	0.001	0.018	0.018	95.2
		2	0.002	0.066	0.064	94.3	0.002	0.026	0.024	93.1
		4	<0.001	0.049	0.051	94.7	<0.001	0.025	0.025	94.5
V. $[-1, 1] \times [-1, 1]$	Gumbel	1.3	<0.001	0.073	0.073	94.0	0.001	0.017	0.017	94.7
		2	<0.001	0.064	0.065	94.5	0.002	0.024	0.024	94.8
		3	-0.002	0.052	0.054	95.5	<0.001	0.026	0.026	93.6
	Frank	1	0.001	0.079	0.075	93.7	<0.001	0.020	0.020	93.8
		4	<0.001	0.068	0.069	94.9	0.003	0.026	0.025	93.8
		7	0.001	0.058	0.060	94.7	0.001	0.030	0.028	92.8

Web Table 4: Comparison of performance metrics for the two estimators under different copula models with sample size $n = 1000$

Region	Model	Parameter	U-statistic-based estimator				Copula model-based estimator				
			Bias	SE _e	ASE _b	CP%	Bias	SE _e	ASE _b	CP%	
I. $(-\infty, -2] \times (-\infty, -2]$	Clayton	1	-0.002	0.140	0.146	94.6	<0.001	0.014	0.014	94.1	
		2	0.001	0.097	0.104	94.9	<0.001	0.011	0.011	94.3	
		4	<0.001	0.071	0.075	94.9	<0.001	0.008	0.008	96.2	
II. $(-\infty, -1] \times (-\infty, -1]$	Clayton	1	0.003	0.064	0.065	95.1	<0.001	0.013	0.014	95.5	
		2	0.002	0.049	0.049	94.5	<0.001	0.011	0.011	95.1	
		4	<0.001	0.032	0.034	96.2	<0.001	0.008	0.008	95.3	
	Gumbel	1.3	0.001	0.095	0.097	94.9	<0.001	0.004	0.004	96.0	
		2	-0.002	0.073	0.074	95.5	<0.001	0.007	0.007	94.9	
		3	-0.003	0.061	0.061	95.3	<0.001	0.008	0.008	95.0	
	Frank	1	-0.003	0.107	0.109	94.6	<0.001	0.001	0.001	95.2	
		4	-0.002	0.078	0.079	95.4	<0.001	0.003	0.003	95.5	
		7	-0.002	0.065	0.068	95.1	<0.001	0.004	0.004	94.0	
III. $[2, +\infty) \times [2, +\infty)$	Gumbel	1.3	-0.005	0.192	0.188	92.4	<0.001	0.009	0.009	94.5	
		2	0.005	0.121	0.127	93.8	<0.001	0.009	0.009	95.6	
		3	-0.004	0.092	0.098	95.0	<0.001	0.008	0.008	95.1	
IV. $[1, +\infty) \times [1, +\infty)$	Clayton	1	<0.001	0.089	0.096	95.2	<0.001	<0.001	<0.001	95.0	
		2	-0.001	0.079	0.083	95.2	<0.001	0.001	0.001	93.8	
		4	-0.003	0.072	0.071	94.6	<0.001	0.002	0.002	93.8	
	Gumbel	1.3	<0.001	0.078	0.081	95.2	<0.001	0.010	0.011	94.8	
		2	<0.001	0.056	0.057	95.2	<0.001	0.009	0.010	95.1	
		3	<0.001	0.044	0.044	94.5	<0.001	0.009	0.008	94.5	
	Frank	1	<0.001	0.105	0.109	95.2	<0.001	0.001	0.001	95.0	
		4	0.002	0.077	0.079	95.2	<0.001	0.003	0.003	94.2	
		7	-0.004	0.067	0.068	94.5	<0.001	0.004	0.004	96.6	
	V. $[-1, 1] \times [-1, 1]$	Clayton	1	<0.001	0.032	0.031	94.3	<0.001	0.008	0.008	95.0
			2	<0.001	0.029	0.028	93.3	<0.001	0.011	0.011	95.2
4			<0.001	0.022	0.022	95.1	<0.001	0.012	0.012	95.1	
Gumbel		1.3	<0.001	0.031	0.032	94.8	<0.001	0.008	0.008	94.6	
		2	<0.001	0.028	0.028	95.6	<0.001	0.010	0.011	95.3	
		3	<0.001	0.024	0.023	94.2	<0.001	0.012	0.011	94.9	
Frank		1	0.002	0.034	0.033	94.3	<0.001	0.009	0.009	95.2	
	4	<0.001	0.030	0.030	95.6	<0.001	0.011	0.011	95.5		
	7	<0.001	0.026	0.026	94.5	<0.001	0.013	0.013	94.2		

Web Table 5: Comparison of performance metrics for the two estimators under different copula models with sample size $n = 2000$

Region	Model	Parameter	U-statistic-based estimator				Copula model-based estimator				
			Bias	SE _e	ASE _b	CP%	Bias	SE _e	ASE _b	CP%	
I. $(-\infty, -2] \times (-\infty, -2]$	Clayton	1	-0.001	0.094	0.096	94.5	<0.001	0.010	0.010	94.5	
		2	<0.001	0.068	0.070	94.4	<0.001	0.008	0.008	95.8	
		4	-0.001	0.049	0.049	93.6	<0.001	0.006	0.005	94.7	
II. $(-\infty, -1] \times (-\infty, -1]$	Clayton	1	-0.002	0.046	0.045	95.5	<0.001	0.009	0.010	95.8	
		2	<0.001	0.015	0.015	96.2	<0.001	0.008	0.008	94.8	
		4	<0.001	0.024	0.024	95.8	<0.001	0.005	0.005	95.3	
	Gumbel	1.3	-0.002	0.065	0.067	95.5	<0.001	0.003	0.003	94.7	
		2	<0.001	0.015	0.015	95.4	<0.001	0.010	0.009	93.5	
		3	<0.001	0.043	0.043	94.8	<0.001	0.006	0.006	94.5	
	Frank	1	<0.001	0.018	0.018	94.1	<0.001	0.001	0.001	94.0	
		4	-0.002	0.055	0.055	94.3	<0.001	0.002	0.002	94.3	
		7	0.002	0.047	0.047	95.5	<0.001	0.003	0.003	94.5	
III. $[2, +\infty) \times [2, +\infty)$	Gumbel	1.3	-0.001	0.124	0.131	95.5	<0.001	0.006	0.006	94.9	
		2	0.003	0.084	0.084	94.5	<0.001	0.006	0.006	94.7	
		3	-0.002	0.064	0.065	93.8	<0.001	0.006	0.006	93.7	
IV. $[1, +\infty) \times [1, +\infty)$	Clayton	1	<0.001	0.064	0.066	95.3	<0.001	<0.001	<0.001	95.4	
		2	-0.002	0.054	0.058	95.5	<0.001	<0.001	<0.001	95.3	
		4	<0.001	0.050	0.051	94.6	<0.001	0.002	0.002	94.7	
	Gumbel	1.3	<0.001	0.054	0.055	95.3	<0.001	0.002	0.002	95.0	
		2	<0.001	0.039	0.040	95.3	<0.001	0.007	0.007	95.4	
		3	0.001	0.031	0.031	94.2	<0.001	0.006	0.006	95.4	
	Frank	1	0.002	0.070	0.070	90.0	<0.001	<0.001	<0.001	93.8	
		4	-0.003	0.055	0.055	95.4	<0.001	0.002	0.002	95.2	
		7	-0.002	0.047	0.047	94.3	<0.001	0.003	0.003	93.6	
	V. $[-1, 1] \times [-1, 1]$	Clayton	1	<0.001	0.022	0.022	95.6	<0.001	0.006	0.006	94.3
			2	<0.001	0.019	0.020	95.4	<0.001	0.008	0.008	95.2
4			<0.001	0.015	0.015	95.9	<0.001	0.009	0.008	94.5	
Gumbel		1.3	<0.001	0.024	0.023	94.3	<0.001	0.005	0.005	94.7	
		2	<0.001	0.020	0.020	93.9	<0.001	0.008	0.008	95.3	
		3	<0.001	0.016	0.016	95.1	<0.001	0.008	0.008	94.5	
Frank		1	<0.001	0.022	0.023	95.9	<0.001	0.006	0.006	94.2	
	4	<0.001	0.022	0.021	94.5	<0.001	0.008	0.008	94.3		
	7	<0.001	0.018	0.018	94.1	<0.001	0.009	0.009	95.1		



Web Figure 15: Comparison of standard errors and 95% coverage probabilities for two estimators in five regions, under different sample sizes generated by various copula models. Horizontal axes: sample size n . Vertical axes: standard errors and 95% coverage probabilities. U-statistic: U-statistic-based estimator. Copula model: Copula model-based estimator. Region I: $(-\infty, -2] \times (-\infty, -2]$. Region II: $(-\infty, -1] \times (-\infty, -1]$. Region III: $[2, +\infty) \times [2, +\infty)$. Region IV: $[1, +\infty) \times [1, +\infty)$. Region V: $[-1, 1] \times [-1, 1]$.

E.2 Analysis of the Underlying Mechanisms for the Superior Performance of the Copula Model-based Estimator

The superior performance of the copula model-based estimator stems from its fundamental differences in information usage and asymptotic efficiency compared to the U-statistic-based estimator, as well as achieving a more favorable balance in the bias-variance trade-off. Specifically,

- 1) **Information Usage.** The U-statistic utilizes only local window samples and their rank information, which leads to limitations in two aspects: (1) the estimation results are sensitive to fluctuations of sample size and outliers; (2) it does not utilize information outside the window, such as the global dependence structure, thereby failing to extract enough information to reduce the estimation uncertainty. These limitations result in its high variance. In contrast, the copula model-based estimator efficiently extracts the global and smooth dependence structure information from the whole dataset through a parametric model fitted globally via maximum likelihood estimation (MLE). Local Kendall's τ is then derived as a continuously differentiable function of the estimated copula parameter. This “global-learning, local-inference” mechanism regularizes the estimator, reducing the uncertainty caused by local data sparsity or outliers.
- 2) **Asymptotic Efficiency.** The copula parameter θ is estimated by MLE. Therefore, under correct model specification, the estimator $\hat{\theta}$ is asymptotically efficient (Casella and Berger, 2002, p. 472), attaining the Cramér-Rao lower bound for asymptotic variance, which means that it achieves the minimum variance among all unbiased estimators. Since our copula-based estimator is a continuously differentiable function of the copula parameter θ , by the delta method (Van der Vaart, 2000), it inherits such asymptotic efficiency (Casella and Berger, 2002). This implies that, under regularity conditions, its asymptotic variance also attains the Cramér-Rao lower bound. In contrast, the U-statistic does not rely on a likelihood function. Even under optimal conditions, its asymptotic variance is strictly greater than the Cramér-Rao lower bound; consequently, it cannot surpass the efficiency of the MLE approach when the model is correctly specified.
- 3) **Bias-Variance Trade-off.** The copula-based estimator achieves a more favorable balance in the classical bias-variance trade-off (Wasserman, 2006). Specifically, although the U-statistic is unbiased, it has to pay the price of a high variance. In contrast, the copula-based estimator, by introducing a negligible small bias (when the model is approximately correct), achieves a significant reduction in variance. This strategy makes the mean squared error of the copula-based estimator significantly smaller than that of the U-statistic, achieving superior overall performance.

For the three reasons mentioned above, the copula-based estimator demonstrates significantly superior performance compared to the U-statistic-based estimator.

E.3 Copula Model Misspecification Experiments

To evaluate the impact of copula model misspecification on the performance of estimators, we conduct a series of simulation studies. Specifically, we set up three types of copula model misspecification scenarios,

- (1) Scenario 1: the true copula model is Clayton copula ($\theta_C = 1, 2, 4$), and the misspecified copula models are Gumbel and Frank copulas.
- (2) Scenario 2: the true copula model is Gumbel copula ($\theta_G = 1.3, 2, 3$), and the misspecified copula models are Clayton and Frank copulas.
- (3) Scenario 3: the true copula model is Frank copula ($\theta_F = 1, 4, 7$), and the misspecified copula models are Clayton and Gumbel copulas.

To ensure more robust results, for each scenario, we use three different copula parameters (from weak to strong) and three different sample sizes ($n = 100, 500, 1000$) to analyze the influence of copula model misspecification on the estimators' five performance metrics in six regions, respectively. The six regions include two left-lower corner regions (Region I: $[0, 0.1] \times [0, 0.1]$; Region II: $[0, 0.2] \times [0, 0.2]$), two right-upper corner regions (Region III: $[0.9, 1] \times (0.9, 1]$; Region IV: $[0.8, 1] \times (0.8, 1]$), and two central regions (Region V: $[0.45, 0.55] \times [0.45, 0.55]$; Region VI: $[0.4, 0.6] \times [0.4, 0.6]$). Five performance metrics are average bias (Bias), root mean square error (RMSE), empirical standard errors (SE_e), average bootstrap standard errors (ASE_b) and empirical 95% coverage probabilities (CP%), respectively.

As expected, all estimators with correctly specified copula models exhibit smaller absolute biases, root mean square errors and standard errors, and their coverage probabilities are very close to the 95% nominal level. This indicates that correctly specifying the copula model can yield unbiased and accurate estimation results across all regions. However, the performance of estimators with misspecified copula models is very poor. Specifically,

For Scenario 1 (see Web Table 6): (1) In the two regions near the lower tail (Regions I and II), the incorrect use of Gumbel and Frank copulas leads to a significant increase in both absolute biases and root mean square errors, and the 95% coverage probabilities are all zero, indicating a complete failure of statistical inference; (2) In the two regions near the upper tail (Regions III and IV), although the misspecified Frank copula yields smaller absolute biases, root mean square errors and standard errors than the Gumbel copula, the 95% coverage probabilities for both are very low (mostly zero), likewise leading to a complete failure of inference; (3) In the two central regions (Regions V and VI), estimators based on misspecified copula models all exhibit small absolute biases, root mean square errors and standard errors, showing relatively good performance. When the dependence is weak ($\theta_C = 1$) and the sample size is small ($n = 100$), their 95% coverage probabilities remain acceptable ($>80\%$). However, as the Clayton copula parameter increases (indicating stronger dependence) or the sample size grows ($n = 500, 1000$), the coverage probabilities decrease significantly. Although they do not drop to zero as in other regions, in most cases they fall below 60%, reaching an unacceptable level. This indicates that even though the estimators perform better in the central regions than in the tail regions, they still exhibit generally low reliability.

For Scenario 2 (see Web Table 7): (1) In the two regions near the lower tail (Regions I and II), although the misspecified Frank copula has smaller absolute biases, root mean

square errors and standard errors than the Clayton copula, the coverage probabilities under both misspecifications are very low (mostly zero or close to zero), making the inference entirely invalid; (2) In the two regions near the upper tail (Regions III and IV), using either Clayton or Frank copula results in a significant increase in both absolute biases and root mean square errors, with coverage probabilities all dropping to zero, leading to statistical inference entirely invalid as well; (3) In the two central regions (Regions V and VI), although the absolute bias, root mean square errors and standard errors for both misspecified Clayton and Frank copulas are very small, the coverage probabilities are generally low (mostly below 60%), and decrease as the Gumbel copula parameter increases (indicating stronger dependence) and the sample size grows ($n = 500, 1000$). This suggests that the estimators are also unreliable in the central regions.

For Scenario 3 (see Web Table 8): (1) In the two regions near the lower tail (Regions I and II), although the absolute biases, root mean square errors and standard errors resulting from the misspecified Gumbel and Clayton copula models are not high, the coverage probabilities are generally below 50% (except when the Frank copula parameter $\theta_F = 1$ and the sample size $n = 100$), or even zero, indicating that the statistical inference is almost invalid; (2) In the two regions near the upper tail (Regions III and IV), copula misspecification similarly leads to very low coverage probabilities (mostly zero or close to zero), rendering inference entirely invalid as well; (3) In the two central regions (Regions V and VI), although the absolute biases, root mean square errors and standard errors for both the Gumbel and Clayton copulas are relatively small, the coverage probabilities remain very low (mostly close to zero), indicating that the estimators' reliability is also poor.

The above results demonstrate that correct specification of the copula model is crucial. Once misspecified, even if the absolute biases, RMSEs and SE_e s are small in some regions, the reliability of statistical inference (as measured by the 95% coverage probability) decreases significantly, leading to invalid estimation. Therefore, in practical applications, researchers should strive to select the copula model correctly based on data characteristics, or employ more robust model selection and diagnostic tools to avoid inference errors caused by model misspecification. Furthermore, prioritizing the use of more flexible copula models (e.g., mixture copulas) can also help enhance the model's adaptability and robustness.

Web Table 6: Comparison of performance metrics for estimators based on misspecified and true copula models under Scenario 1

The true copula model Clayton copula			The correctly specified copula model Clayton copula					The misspecified copula model 1 Gumbel copula					The misspecified copula model 2 Frank copula					
Region	Parameter	Sample size	Bias	RMSE	SE _e	ASE _b	CP%	Bias	RMSE	SE _e	ASE _b	CP%	Bias	RMSE	SE _e	ASE _b	CP%	
I	1	100	0.0029	0.0438	0.0437	0.0433	92.8	-0.2928	0.2931	0.0135	0.0138	0.0	-0.3232	0.3233	0.0033	0.0033	0.0	
		500	0.0014	0.0192	0.0192	0.0193	93.8	-0.2939	0.2940	0.0062	0.0061	0.0	-0.3236	0.3236	0.0014	0.0014	0.0	
		1000	0.0003	0.0139	0.0140	0.0137	93.8	-0.2938	0.2939	0.0043	0.0043	0.0	-0.3236	0.3236	0.0010	0.0010	0.0	
	2	100	0.0001	0.0327	0.0327	0.0344	95.4	-0.4222	0.4226	0.0189	0.0196	0.0	-0.4763	0.4763	0.0050	0.0055	0.0	
		500	0.0005	0.0158	0.0158	0.0154	93.2	-0.4227	0.4228	0.0084	0.0087	0.0	-0.4766	0.4766	0.0023	0.0024	0.0	
		1000	0.0000	0.0115	0.0115	0.0110	93.6	-0.4231	0.4232	0.0062	0.0061	0.0	-0.4767	0.4767	0.0017	0.0017	0.0	
	4	100	0.0010	0.0257	0.0257	0.0241	92.2	-0.5238	0.5246	0.0294	0.0293	0.0	-0.6105	0.6106	0.0105	0.0102	0.0	
		500	0.0003	0.0109	0.0109	0.0109	94.8	-0.5224	0.5271	0.0129	0.0131	0.0	-0.5269	0.5271	0.0129	0.0131	0.0	
		1000	-0.0003	0.0080	0.0080	0.0078	94.4	-0.5268	0.5269	0.0096	0.0093	0.0	-0.6119	0.6119	0.0032	0.0031	0.0	
	II	1	100	0.0043	0.0436	0.0435	0.0432	93.0	-0.2793	0.2798	0.0179	0.0178	0.0	-0.3007	0.3008	0.0095	0.0097	0.0
			500	0.0001	0.0197	0.0197	0.0194	95.0	-0.2818	0.2819	0.0078	0.0079	0.0	-0.3019	0.3019	0.0041	0.0042	0.0
			1000	0.0000	0.0138	0.0138	0.0138	93.2	-0.2816	0.2817	0.0058	0.0055	0.0	-0.3019	0.3020	0.0031	0.0030	0.0
2		100	0.0023	0.0343	0.0343	0.0338	93.6	-0.3956	0.3964	0.0250	0.0244	0.0	-0.4290	0.4292	0.0144	0.0143	0.0	
		500	0.0004	0.0156	0.0156	0.0154	95.0	-0.3995	0.3997	0.0112	0.0109	0.0	-0.4307	0.4308	0.0063	0.0063	0.0	
		1000	-0.0001	0.0114	0.0114	0.0109	94.0	-0.3996	0.3997	0.0078	0.0077	0.0	-0.4309	0.4310	0.0046	0.0044	0.0	
4		100	-0.0003	0.0240	0.0240	0.0240	94.0	-0.4858	0.4870	0.0339	0.0348	0.0	-0.5188	0.5193	0.0221	0.0228	0.0	
		500	-0.0001	0.0108	0.0108	0.0109	94.8	-0.4897	0.4900	0.0151	0.0159	0.0	-0.5208	0.5209	0.0096	0.0106	0.0	
		1000	0.0000	0.0078	0.0078	0.0078	94.4	-0.4891	0.4892	0.0114	0.0112	0.0	-0.5202	0.5202	0.0072	0.0072	0.0	
III		1	100	0.0001	0.0006	0.0006	0.0006	94.0	0.2660	0.2683	0.0351	0.0382	0.0	0.0082	0.0088	0.0032	0.0033	20.4
			500	0.0000	0.0002	0.0002	0.0002	95.8	0.2663	0.2667	0.0151	0.0158	0.0	0.0080	0.0081	0.0014	0.0014	0.0
			1000	0.0000	0.0002	0.0002	0.0002	95.4	0.2674	0.2676	0.0111	0.0111	0.0	0.0081	0.0081	0.0010	0.0010	0.0
	2	100	0.0001	0.0012	0.0012	0.0012	94.6	0.3491	0.3509	0.0359	0.0362	0.0	0.0188	0.0195	0.0053	0.0055	0.0	
		500	0.0000	0.0005	0.0005	0.0005	95.8	0.3459	0.3462	0.0149	0.0163	0.0	0.0182	0.0183	0.0021	0.0024	0.0	
		1000	0.0000	0.0004	0.0004	0.0004	93.6	0.3457	0.3459	0.0113	0.0116	0.0	0.0183	0.0183	0.0017	0.0017	0.0	
	4	100	0.0003	0.0029	0.0028	0.0029	95.0	0.4420	0.4442	0.0440	0.0399	0.0	0.0407	0.0420	0.0105	0.0101	0.0	
		500	0.0000	0.0012	0.0012	0.0012	96.6	0.4387	0.4390	0.0179	0.0184	0.0	0.0393	0.0395	0.0042	0.0048	0.0	
		1000	0.0000	0.0009	0.0009	0.0009	93.0	0.4382	0.4384	0.0130	0.0131	0.0	0.0392	0.0394	0.0032	0.0031	0.0	
	IV	1	100	0.0004	0.0021	0.0021	0.0021	94.6	0.2525	0.2557	0.0403	0.0420	0.6	0.0256	0.0274	0.0097	0.0095	15.0
			500	0.0000	0.0009	0.0009	0.0009	95.0	0.2517	0.2522	0.0166	0.0179	0.0	0.0245	0.0249	0.0040	0.0042	0.0
			1000	0.0001	0.0006	0.0006	0.0006	96.0	0.2525	0.2528	0.0122	0.0126	0.0	0.0246	0.0248	0.0028	0.0030	0.0
2		100	0.0006	0.0040	0.0040	0.0042	95.6	0.3306	0.3328	0.0380	0.0391	0.0	0.0513	0.0533	0.0144	0.0144	0.6	
		500	0.0001	0.0018	0.0018	0.0018	94.8	0.3295	0.3299	0.0166	0.0175	0.0	0.0500	0.0504	0.0063	0.0063	0.0	
		1000	0.0001	0.0013	0.0013	0.0013	95.2	0.3294	0.3296	0.0122	0.0123	0.0	0.0498	0.0500	0.0044	0.0044	0.0	
4		100	0.0010	0.0083	0.0083	0.0086	95.6	0.4052	0.4072	0.0412	0.0416	0.0	0.0957	0.0983	0.0223	0.0229	0.0	
		500	0.0003	0.0037	0.0037	0.0038	95.2	0.4023	0.4028	0.0194	0.0191	0.0	0.0933	0.0938	0.0101	0.0107	0.0	
		1000	0.0000	0.0025	0.0025	0.0026	96.0	0.4035	0.4037	0.0135	0.0135	0.0	0.0932	0.0935	0.0070	0.0071	0.0	
V		1	100	0.0001	0.0007	0.0007	0.0007	95.4	-0.0005	0.0010	0.0009	0.0009	83.9	0.0017	0.0021	0.0013	0.0013	85.9
			500	0.0000	0.0003	0.0003	0.0003	94.9	-0.0006	0.0007	0.0004	0.0004	58.9	0.0015	0.0016	0.0005	0.0005	15.8
			1000	0.0000	0.0002	0.0002	0.0002	95.3	-0.0006	0.0007	0.0003	0.0003	36.4	0.0015	0.0016	0.0004	0.0004	0.9
	2	100	0.0002	0.0014	0.0014	0.0015	95.3	-0.0022	0.0026	0.0014	0.0015	57.9	0.0030	0.0038	0.0023	0.0025	89.8	
		500	0.0001	0.0006	0.0006	0.0006	95.6	-0.0024	0.0025	0.0006	0.0006	5.1	0.0028	0.0030	0.0010	0.0010	19.4	
		1000	0.0000	0.0005	0.0005	0.0004	93.4	-0.0024	0.0025	0.0004	0.0004	0.5	0.0027	0.0028	0.0007	0.0007	1.5	
	4	100	0.0005	0.0041	0.0040	0.0041	95.1	-0.0101	0.0106	0.0032	0.0031	14.8	0.0062	0.0088	0.0062	0.0061	90.9	
		500	0.0001	0.0017	0.0017	0.0017	95.1	-0.0108	0.0109	0.0013	0.0013	0.0	0.0053	0.0059	0.0025	0.0030	56.6	
		1000	0.0001	0.0012	0.0012	0.0012	95.7	-0.0108	0.0108	0.0009	0.0009	0.0	0.0054	0.0057	0.0018	0.0018	14.0	
	VI	1	100	0.0003	0.0028	0.0028	0.0029	95.0	-0.0021	0.0040	0.0034	0.0034	84.3	0.0065	0.0081	0.0048	0.0050	84.5
			500	0.0001	0.0013	0.0013	0.0013	93.2	-0.0024	0.0028	0.0015	0.0015	60.1	0.0060	0.0064	0.0022	0.0021	15.6
			1000	0.0000	0.0009	0.0009	0.0009	94.7	-0.0025	0.0027	0.0010	0.0010	32.3	0.0058	0.0060	0.0015	0.0015	1.2
2		100	0.0006	0.0054	0.0054	0.0057	96.1	-0.0089	0.0105	0.0055	0.0056	57.3	0.0107	0.0136	0.0084	0.0089	88.1	
		500	0.0001	0.0024	0.0024	0.0024	94.9	-0.0095	0.0098	0.0024	0.0024	4.2	0.0099	0.0106	0.0038	0.0038	22.3	
		1000	0.0000	0.0017	0.0017	0.0017	95.2	-0.0095	0.0097	0.0017	0.0017	0.0	0.0099	0.0103	0.0027	0.0027	1.9	
4		100	0.0014	0.0136	0.0135	0.0136	94.3	-0.0376	0.0392	0.0109	0.0112	13.8	0.0175	0.0254	0.0185	0.0191	92.0	
		500	0.0004	0.0059	0.0059	0.0059	94.7	-0.0395	0.0397	0.0047	0.0046	0.0	0.0159	0.0180	0.0084	0.0092	60.9	
		1000	0.0001	0.0044	0.0044	0.0042	93.6	-0.0398	0.0399	0.0033	0.0033	0.0	0.0153	0.0165	0.0059	0.0058	24.3	

Note: The original results are stored in local dependence toolbox/simulation_study/copula-misspec/experiment_results.xlsx.

Web Table 7: Comparison of performance metrics for estimators based on misspecified and true copula models under Scenario 2

Region	The true copula model			The correctly specified copula model					The misspecified copula model 1					The misspecified copula model 2				
	Gumbel copula			Gumbel copula					Clayton copula					Frank copula				
	Parameter	Sample size	Bias	RMSE	SE _e	ASE _b	CP%		Bias	RMSE	SE _e	ASE _b	CP%	Bias	RMSE	SE _e	ASE _b	CP%
I	1.3	100	-0.0003	0.0106	0.0106	0.0106	93.0		0.1151	0.1302	0.0609	0.0556	46.4	-0.0251	0.0252	0.0023	0.0023	0.0
		500	0.0004	0.0046	0.0046	0.0047	95.4		0.1178	0.1204	0.0251	0.0258	0.2	-0.0251	0.0251	0.0010	0.0010	0.0
		1000	0.0000	0.0033	0.0033	0.0033	94.4		0.1181	0.1194	0.0178	0.0183	0.0	-0.0251	0.0251	0.0007	0.0007	0.0
	2	100	0.0018	0.0189	0.0189	0.0181	92.0		0.2383	0.2443	0.0537	0.0517	1.0	-0.0839	0.0841	0.0057	0.0056	0.0
		500	0.0004	0.0079	0.0079	0.0081	96.0		0.2376	0.2389	0.0250	0.0240	0.0	-0.0845	0.0845	0.0025	0.0024	0.0
		1000	0.0000	0.0056	0.0056	0.0057	95.0		0.2349	0.2354	0.0156	0.0170	0.0	-0.0847	0.0847	0.0016	0.0017	0.0
	3	100	0.0036	0.0245	0.0243	0.0235	94.4		0.2864	0.2908	0.0507	0.0474	0.4	-0.1546	0.1550	0.0100	0.0099	0.0
		500	0.0013	0.0110	0.0110	0.0106	93.8		0.2849	0.2857	0.0225	0.0221	0.0	-0.1557	0.1557	0.0045	0.0049	0.0
		1000	0.0008	0.0077	0.0076	0.0075	94.4		0.2861	0.2865	0.0151	0.0156	0.0	-0.1560	0.1561	0.0031	0.0030	0.0
	1.3	100	0.0019	0.0142	0.0140	0.0138	93.4		0.1114	0.1258	0.0585	0.0550	49.4	-0.0216	0.0228	0.0073	0.0074	22.8
		500	0.0000	0.0061	0.0061	0.0061	95.2		0.1048	0.1079	0.0258	0.0256	1.2	-0.0227	0.0229	0.0031	0.0032	0.0
		1000	-0.0001	0.0044	0.0044	0.0043	94.4		0.1061	0.1078	0.0191	0.0182	0.0	-0.0228	0.0229	0.0023	0.0023	0.0
II	2	100	0.0022	0.0232	0.0231	0.0223	92.8		0.2101	0.2174	0.0556	0.0512	4.6	-0.0672	0.0686	0.0142	0.0143	1.4
		500	0.0002	0.0097	0.0097	0.0100	96.8		0.2051	0.2066	0.0243	0.0239	0.0	-0.0688	0.0691	0.0063	0.0063	0.0
		1000	-0.0004	0.0067	0.0067	0.0070	95.6		0.2049	0.2056	0.0161	0.0170	0.0	-0.0691	0.0692	0.0043	0.0044	0.0
	3	100	0.0017	0.0284	0.0284	0.0270	92.4		0.2321	0.2374	0.0499	0.0477	1.4	-0.1124	0.1148	0.0232	0.0219	1.6
		500	0.0015	0.0127	0.0126	0.0123	94.4		0.2349	0.2359	0.0221	0.0220	0.0	-0.1136	0.1140	0.0099	0.0107	0.2
		1000	-0.0002	0.0089	0.0089	0.0086	94.2		0.2326	0.2331	0.0160	0.0157	0.0	-0.1145	0.1147	0.0070	0.0069	0.0
	1.3	100	-0.0008	0.0319	0.0319	0.0352	97.4		-0.2435	0.2435	0.0002	0.0003	0.0	-0.2386	0.2386	0.0023	0.0024	0.0
		500	0.0000	0.0142	0.0142	0.0138	93.6		-0.2435	0.2435	0.0001	0.0001	0.0	-0.2389	0.2389	0.0010	0.0010	0.0
		1000	-0.0003	0.0098	0.0098	0.0098	95.6		-0.2435	0.2435	0.0001	0.0001	0.0	-0.2389	0.2389	0.0007	0.0007	0.0
	2	100	0.0022	0.0288	0.0287	0.0288	94.0		-0.4039	0.4039	0.0008	0.0008	0.0	-0.3817	0.3817	0.0056	0.0055	0.0
		500	0.0004	0.0127	0.0127	0.0130	96.2		-0.4041	0.4041	0.0003	0.0003	0.0	-0.3822	0.3822	0.0024	0.0024	0.0
		1000	0.0000	0.0093	0.0094	0.0092	94.0		-0.4041	0.4041	0.0002	0.0002	0.0	-0.3823	0.3823	0.0017	0.0017	0.0
III	3	100	0.0023	0.0267	0.0266	0.0261	94.2		-0.5385	0.5385	0.0017	0.0018	0.0	-0.4870	0.4871	0.0099	0.0099	0.0
		500	0.0012	0.0115	0.0114	0.0118	96.0		-0.5388	0.5388	0.0007	0.0007	0.0	-0.4877	0.4877	0.0043	0.0048	0.0
		1000	-0.0002	0.0084	0.0084	0.0083	94.2		-0.5389	0.5389	0.0005	0.0005	0.0	-0.4882	0.4882	0.0031	0.0030	0.0
	1.3	100	-0.0006	0.0396	0.0396	0.0389	96.6		-0.2290	0.2290	0.0011	0.0011	0.0	-0.2126	0.2127	0.0078	0.0074	0.0
		500	-0.0004	0.0163	0.0163	0.0161	94.4		-0.2293	0.2293	0.0004	0.0004	0.0	-0.2136	0.2137	0.0034	0.0032	0.0
		1000	-0.0009	0.0109	0.0108	0.0113	95.8		-0.2293	0.2293	0.0003	0.0003	0.0	-0.2138	0.2138	0.0021	0.0023	0.0
	2	100	0.0027	0.0307	0.0306	0.0302	94.6		-0.3990	0.3990	0.0028	0.0029	0.0	-0.3350	0.3354	0.0145	0.0144	0.0
		500	0.0010	0.0136	0.0136	0.0135	94.6		-0.3993	0.3993	0.0012	0.0012	0.0	-0.3364	0.3365	0.0066	0.0063	0.0
		1000	-0.0009	0.0103	0.0102	0.0096	94.2		-0.3995	0.3995	0.0009	0.0008	0.0	-0.3373	0.3374	0.0048	0.0045	0.0
	3	100	0.0017	0.0284	0.0284	0.0263	93.6		-0.5288	0.5288	0.0061	0.0060	0.0	-0.3974	0.3981	0.0240	0.0221	0.0
		500	0.0005	0.0119	0.0119	0.0120	95.0		-0.5298	0.5298	0.0025	0.0025	0.0	-0.4001	0.4002	0.0097	0.0105	0.0
		1000	0.0005	0.0082	0.0082	0.0085	95.2		-0.5300	0.5300	0.0017	0.0018	0.0	-0.4004	0.4005	0.0067	0.0069	0.0
IV	1.3	100	0.0001	0.0006	0.0006	0.0007	93.2		-0.0009	0.0010	0.0005	0.0005	50.6	0.0009	0.0013	0.0009	0.0010	92.0
		500	0.0000	0.0003	0.0003	0.0003	94.4		-0.0009	0.0009	0.0002	0.0002	2.4	0.0009	0.0010	0.0004	0.0004	46.8
		1000	0.0000	0.0002	0.0002	0.0002	95.0		-0.0009	0.0009	0.0002	0.0001	0.0	0.0009	0.0009	0.0003	0.0003	14.2
	2	100	0.0002	0.0015	0.0015	0.0015	96.0		-0.0038	0.0040	0.0011	0.0010	12.8	0.0034	0.0042	0.0025	0.0025	84.4
		500	0.0000	0.0006	0.0006	0.0007	94.8		-0.0040	0.0040	0.0004	0.0004	0.0	0.0030	0.0031	0.0010	0.0010	13.6
		1000	0.0000	0.0005	0.0005	0.0005	95.0		-0.0040	0.0040	0.0003	0.0003	0.0	0.0030	0.0031	0.0007	0.0007	0.6
	3	100	0.0004	0.0032	0.0032	0.0033	94.2		-0.0106	0.0108	0.0020	0.0022	3.4	0.0093	0.0107	0.0054	0.0059	75.4
		500	0.0001	0.0015	0.0015	0.0014	94.2		-0.0110	0.0110	0.0009	0.0009	0.0	0.0086	0.0090	0.0026	0.0028	7.4
		1000	0.0000	0.0010	0.0010	0.0010	96.2		-0.0110	0.0111	0.0006	0.0006	0.0	0.0085	0.0087	0.0018	0.0018	0.0
	1.3	100	0.0002	0.0026	0.0026	0.0026	94.4		-0.0034	0.0039	0.0019	0.0019	50.2	0.0036	0.0053	0.0039	0.0039	91.6
		500	0.0001	0.0012	0.0011	0.0011	93.6		-0.0036	0.0037	0.0008	0.0008	2.0	0.0036	0.0039	0.0017	0.0017	39.8
		1000	0.0001	0.0008	0.0008	0.0008	93.6		-0.0036	0.0037	0.0006	0.0006	0.0	0.0036	0.0038	0.0012	0.0012	12.2
V	2	100	0.0007	0.0056	0.0056	0.0058	95.2		-0.0148	0.0153	0.0041	0.0039	9.6	0.0121	0.0151	0.0091	0.0091	84.6
		500	0.0003	0.0024	0.0024	0.0025	96.6		-0.0153	0.0154	0.0017	0.0016	0.0	0.0114	0.0120	0.0038	0.0039	11.8
		1000	0.0001	0.0017	0.0017	0.0018	95.0		-0.0154	0.0155	0.0011	0.0012	0.0	0.0110	0.0113	0.0026	0.0027	0.2
	3	100	0.0012	0.0117	0.0117	0.0114	93.4		-0.0381	0.0388	0.0075	0.0083	3.6	0.0291	0.0340	0.0176	0.0184	69.4
		500	0.0003	0.0049	0.0049	0.0050	95.0		-0.0396	0.0397	0.0034	0.0034	0.0	0.0279	0.0290	0.0079	0.0088	5.6
		1000	0.0002	0.0036	0.0036	0.0035	95.2		-0.0398	0.0398	0.0025	0.0024	0.0	0.0277	0.0282	0.0058	0.0057	0.0

Note: The original results are stored in local dependence toolbox/simulation_study/copula-misspec/experiment_results.xlsx.

Web Table 8: Comparison of performance metrics for estimators based on misspecified and true copula models under Scenario 3

The true copula model			The correctly specified copula model					The misspecified copula model 1					The misspecified copula model 3					
Frank copula			Frank copula					Gumbel copula					Clayton copula					
Region	Parameter	Sample size	Bias	RMSE	SE _e	ASE _b	CP%	Bias	RMSE	SE _e	ASE _b	CP%	Bias	RMSE	SE _e	ASE _b	CP%	
I	1	100	0.0003	0.0014	0.0013	0.0014	90.0	0.0080	0.0107	0.0071	0.0071	91.4	0.0764	0.0930	0.0530	0.0495	72.8	
		500	0.0001	0.0006	0.0006	0.0006	93.6	0.0071	0.0078	0.0033	0.0033	45.4	0.0724	0.0766	0.0250	0.0251	18.0	
		1000	0.0000	0.0004	0.0004	0.0004	95.8	0.0069	0.0072	0.0023	0.0023	13.2	0.0706	0.0728	0.0178	0.0178	1.2	
	4	100	0.0002	0.0039	0.0039	0.0037	92.2	0.0394	0.0421	0.0147	0.0137	14.0	0.2557	0.2615	0.0547	0.0534	0.8	
		500	0.0001	0.0017	0.0017	0.0017	95.4	0.0392	0.0397	0.0063	0.0062	0.0	0.2560	0.2572	0.0254	0.0249	0.0	
		1000	0.0001	0.0012	0.0012	0.0012	94.0	0.0387	0.0389	0.0043	0.0043	0.0	0.2554	0.2560	0.0176	0.0178	0.0	
	7	100	0.0007	0.0064	0.0064	0.0065	94.0	0.0727	0.0755	0.0206	0.0199	3.0	0.3672	0.3714	0.0558	0.0534	0.0	
		500	0.0002	0.0030	0.0029	0.0028	93.2	0.0717	0.0723	0.0090	0.0090	0.0	0.3650	0.3659	0.0257	0.0254	0.0	
		1000	0.0000	0.0020	0.0020	0.0020	94.8	0.0710	0.0713	0.0065	0.0063	0.0	0.3635	0.3640	0.0184	0.0180	0.0	
	II	1	100	0.0007	0.0047	0.0047	0.0046	92.0	0.0063	0.0115	0.0096	0.0094	94.0	0.0722	0.0881	0.0505	0.0502	74.6
			500	0.0002	0.0021	0.0021	0.0020	93.4	0.0056	0.0072	0.0046	0.0044	79.2	0.0656	0.0705	0.0260	0.0250	26.6
			1000	0.0001	0.0014	0.0014	0.0014	95.8	0.0054	0.0063	0.0031	0.0031	62.4	0.0672	0.0696	0.0183	0.0180	4.2
4		100	0.0008	0.0102	0.0102	0.0107	94.6	0.0276	0.0327	0.0176	0.0179	71.2	0.2324	0.2388	0.0550	0.0543	1.8	
		500	0.0001	0.0046	0.0046	0.0047	94.8	0.0267	0.0277	0.0077	0.0078	5.6	0.2273	0.2285	0.0242	0.0250	0.0	
		1000	0.0000	0.0033	0.0033	0.0033	95.0	0.0263	0.0268	0.0056	0.0056	0.0	0.2277	0.2284	0.0173	0.0177	0.0	
7		100	0.0006	0.0154	0.0154	0.0159	96.0	0.0409	0.0475	0.0242	0.0245	62.0	0.3037	0.3090	0.0570	0.0545	0.4	
		500	0.0001	0.0070	0.0070	0.0071	95.2	0.0406	0.0420	0.0109	0.0111	4.0	0.3048	0.3058	0.0251	0.0251	0.0	
		1000	-0.0001	0.0051	0.0051	0.0050	94.2	0.0406	0.0414	0.0081	0.0078	0.0	0.3043	0.3049	0.0185	0.0179	0.0	
III		1	100	0.0002	0.0013	0.0013	0.0014	91.6	0.1319	0.1480	0.0673	0.0589	41.4	-0.0014	0.0014	0.0002	0.0002	0.2
			500	0.0000	0.0006	0.0006	0.0006	94.6	0.1454	0.1486	0.0306	0.0304	6.2	-0.0015	0.0015	0.0001	0.0001	0.0
			1000	0.0000	0.0004	0.0004	0.0004	95.4	0.1481	0.1492	0.0184	0.0203	0.2	-0.0015	0.0015	0.0000	0.0000	0.0
	4	100	0.0005	0.0037	0.0037	0.0038	95.6	0.2883	0.2901	0.0320	0.0315	0.0	-0.0119	0.0119	0.0005	0.0005	0.0	
		500	0.0000	0.0016	0.0016	0.0016	96.2	0.2855	0.2858	0.0137	0.0140	0.0	-0.0120	0.0120	0.0002	0.0002	0.0	
		1000	0.0001	0.0012	0.0012	0.0012	93.8	0.2859	0.2860	0.0098	0.0099	0.0	-0.0120	0.0120	0.0001	0.0001	0.0	
	7	100	0.0008	0.0066	0.0066	0.0065	94.4	0.3691	0.3705	0.0323	0.0322	0.0	-0.0297	0.0297	0.0011	0.0011	0.0	
		500	0.0001	0.0026	0.0026	0.0028	96.0	0.3665	0.3667	0.0139	0.0147	0.0	-0.0298	0.0298	0.0005	0.0005	0.0	
		1000	0.0001	0.0019	0.0019	0.0020	97.6	0.3660	0.3661	0.0103	0.0105	0.0	-0.0299	0.0299	0.0003	0.0003	0.0	
	IV	1	100	0.0008	0.0047	0.0046	0.0047	90.6	0.1053	0.1224	0.0623	0.0555	55.6	-0.0052	0.0052	0.0006	0.0007	0.2
			500	0.0002	0.0021	0.0020	0.0020	94.2	0.1121	0.1159	0.0294	0.0303	10.0	-0.0054	0.0054	0.0003	0.0003	0.0
			1000	0.0001	0.0014	0.0014	0.0014	95.2	0.1125	0.1143	0.0203	0.0214	1.0	-0.0054	0.0054	0.0002	0.0002	0.0
4		100	0.0014	0.0104	0.0103	0.0107	96.0	0.2540	0.2564	0.0352	0.0347	0.0	-0.0363	0.0364	0.0019	0.0020	0.0	
		500	0.0000	0.0045	0.0046	0.0047	96.5	0.2512	0.2517	0.0150	0.0154	0.0	-0.0367	0.0367	0.0008	0.0008	0.0	
		1000	0.0003	0.0034	0.0034	0.0033	94.0	0.2517	0.2519	0.0108	0.0109	0.0	-0.0367	0.0367	0.0006	0.0006	0.0	
7		100	0.0019	0.0165	0.0164	0.0161	94.2	0.3095	0.3113	0.0338	0.0337	0.0	-0.0817	0.0818	0.0040	0.0040	0.0	
		500	0.0002	0.0065	0.0065	0.0071	96.0	0.3068	0.3072	0.0146	0.0154	0.0	-0.0823	0.0823	0.0017	0.0017	0.0	
		1000	0.0000	0.0047	0.0047	0.0050	97.4	0.3063	0.3065	0.0108	0.0110	0.0	-0.0823	0.0823	0.0012	0.0012	0.0	
V		1	100	0.0001	0.0007	0.0007	0.0007	93.8	-0.0006	0.0007	0.0004	0.0004	63.2	-0.0007	0.0008	0.0003	0.0003	42.8
			500	0.0000	0.0003	0.0003	0.0003	96.0	-0.0006	0.0007	0.0002	0.0002	14.8	-0.0008	0.0008	0.0002	0.0001	1.0
			1000	0.0000	0.0002	0.0002	0.0002	95.8	-0.0006	0.0006	0.0001	0.0001	1.8	-0.0008	0.0008	0.0001	0.0001	0.0
	4	100	0.0001	0.0014	0.0014	0.0015	95.8	-0.0026	0.0028	0.0009	0.0009	20.4	-0.0036	0.0037	0.0007	0.0007	2.6	
		500	0.0000	0.0007	0.0007	0.0006	93.8	-0.0026	0.0027	0.0004	0.0004	0.2	-0.0037	0.0037	0.0003	0.0003	0.0	
		1000	0.0000	0.0004	0.0004	0.0005	97.2	-0.0027	0.0027	0.0003	0.0003	0.0	-0.0037	0.0037	0.0002	0.0002	0.0	
	7	100	0.0005	0.0032	0.0032	0.0032	94.2	-0.0070	0.0072	0.0017	0.0017	7.0	-0.0097	0.0098	0.0014	0.0014	1.4	
		500	0.0001	0.0014	0.0013	0.0014	95.4	-0.0073	0.0074	0.0007	0.0007	0.0	-0.0100	0.0100	0.0006	0.0006	0.0	
		1000	0.0000	0.0010	0.0010	0.0010	94.8	-0.0073	0.0073	0.0005	0.0005	0.0	-0.0100	0.0100	0.0004	0.0004	0.0	
	VI	1	100	0.0002	0.0029	0.0028	0.0028	90.2	-0.0023	0.0029	0.0017	0.0017	63.6	-0.0027	0.0031	0.0014	0.0014	46.0
			500	0.0001	0.0013	0.0013	0.0013	94.8	-0.0025	0.0026	0.0008	0.0008	15.6	-0.0030	0.0031	0.0006	0.0006	1.2
			1000	0.0000	0.0009	0.0009	0.0009	95.0	-0.0025	0.0026	0.0005	0.0006	1.6	-0.0030	0.0030	0.0004	0.0004	0.0
4		100	0.0005	0.0055	0.0055	0.0058	96.4	-0.0102	0.0107	0.0034	0.0036	22.8	-0.0140	0.0142	0.0028	0.0029	3.6	
		500	0.0002	0.0024	0.0023	0.0025	95.4	-0.0103	0.0104	0.0015	0.0015	0.0	-0.0143	0.0144	0.0012	0.0012	0.0	
		1000	0.0001	0.0018	0.0018	0.0017	95.8	-0.0103	0.0103	0.0011	0.0011	0.0	-0.0144	0.0144	0.0009	0.0009	0.0	
7		100	0.0020	0.0112	0.0111	0.0113	94.6	-0.0259	0.0266	0.0062	0.0064	5.6	-0.0359	0.0363	0.0051	0.0054	0.4	
		500	0.0001	0.0049	0.0049	0.0048	94.0	-0.0268	0.0270	0.0027	0.0027	0.0	-0.0368	0.0368	0.0023	0.0022	0.0	
		1000	0.0000	0.0034	0.0034	0.0034	93.6	-0.0269	0.0270	0.0019	0.0019	0.0	-0.0369	0.0369	0.0016	0.0016	0.0	

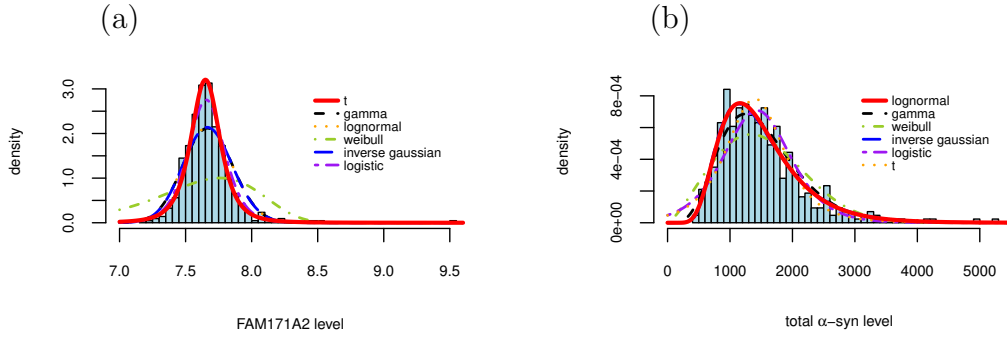
Note: The original results are stored in local dependence toolbox/simulation_study/copula-misspec/experiment_results.xlsx.

Web Appendix F Additional Results for Two Data Examples

This section supplements some results in two real data examples, and introduces the used seven Archimedean mixture copula models.

F.1 Example 1

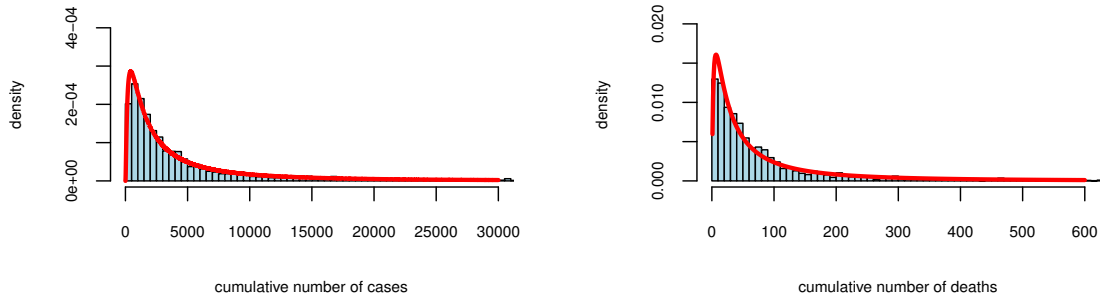
Web Figure 16(a) and (b) display the fitting results of different distribution functions to CSF FAM171A2 and total α -syn in Example 1, respectively.



Web Figure 16: Distribution fitting results for Example 1. (a) Left plot: CSF FAM171A2 level; (b) Right plot: total α -syn level.

F.2 Example 2

Web Figure 17(a) and (b) illustrate the fitting results of lognormal-Pareto distribution to cumulative number of COVID-19 cases and deaths in Example 2, respectively.



Web Figure 17: Distribution fitting results for Example 2. (a) Left plot: cumulative number of cases; (b) Right plot: cumulative number of deaths.

F.3 Archimedean Mixture Copula Models

Web Table 9 lists the seven Archimedean mixture copula models used in the two examples.

Web Table 9: Archimedean mixture copula models used in the data examples

	Copula models	Expressions
1	$MixGC(u, v; \omega_G, \delta_G, \theta_C)$	$\omega_G C_{Gumbel}(u, v; \delta_G) + (1 - \omega_G) C_{Clayton}(u, v; \theta_C)$
2	$MixrG180rC180(u, v; \omega_{rG180}, \delta_{rG180}, \theta_{rC180})$	$\omega_{rG180} C_{rotated\ Gumbel\ 180^\circ}(u, v; \delta_{rG180}) + (1 - \omega_{rG180}) C_{rotated\ Clayton\ 180^\circ}(u, v; \theta_{rC180})$
3	$MixGrG180(u, v; \omega_G, \delta_G, \delta_{rG180})$	$\omega_G C_{Gumbel}(u, v; \delta_G) + (1 - \omega_G) C_{rotated\ Gumbel\ 180^\circ}(u, v; \delta_{rG180})$
4	$MixCrC180(u, v; \omega_C, \theta_C, \theta_{rC180})$	$\omega_C C_{Clayton}(u, v; \theta_C) + (1 - \omega_C) C_{rotated\ Clayton\ 180^\circ}(u, v; \theta_{rC180})$
5	$MixFC(u, v; \omega_F, \lambda_F, \theta_C)$	$\omega_F C_{Frank}(u, v; \lambda_F) + (1 - \omega_F) C_{Clayton}(u, v; \theta_C)$
6	$MixFrC180(u, v; \omega_F, \lambda_F, \theta_{rC180})$	$\omega_F C_{Frank}(u, v; \lambda_F) + (1 - \omega_F) C_{rotated\ Clayton\ 180^\circ}(u, v; \theta_{rC180})$
7	$MixFG(u, v; \omega_F, \lambda_F, \delta_G)$	$\omega_F C_{Frank}(u, v; \lambda_F) + (1 - \omega_F) C_{Gumbel}(u, v; \delta_G)$

The rotated Gumbel copula 180° : $C_{rotated\ Gumbel\ 180^\circ}(u, v; \delta_{rG180}) = u + v - 1 + C_{Gumbel}(1 - u, 1 - v; \delta_{rG180})$; the rotated Clayton copula 180° : $C_{rotated\ Clayton\ 180^\circ}(u, v; \theta_{rC180}) = u + v - 1 + C_{Clayton}(1 - u, 1 - v; \theta_{rC180})$. ω_* is the weight parameter of the mixture copula model, which lies in $[0, 1]$.

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