Copula-based Local Dependence Framework

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Abstract

This paper proposes a novel copula-based local Kendall's tau framework to uncover richer nonlinear local dependence between two financial return series. This framework nests the concepts of global dependence, tail dependence and local dependence. Closed form solutions of local Kendall's tau in terms of copula link local dependence with their global dependence structure together, providing a generalized framework for investigating dependence between two return series. We further extend the copula-based local dependence framework to Spearman's rho. Using this framework, we draw the local Kendall's tau surfaces in different quadrants for some common used bivariate Archimedean copulas. Finally, we demonstrate the advantages of copula-based local Kendall's tau relative to global Kendall's tau with stock market data.

JEL classification: C13; C58; G15

Keywords: Copulas; Local Dependence Measures; Local Dependence Surface; Tail Dependence

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1 Introduction

Dependence measures are widely used in statistics and finance. One of the most important applications of dependence measure is studying the relationship between different financial return series (see, for example, Forbes and Rigobon, 2002; Rodriguez, 2007; Okimoto, 2008; Asimit et al., 2016), which is helpful for financial risk management. The rank dependence coefficients, e.g., Kendall's tau (Kendall, 1938) & Spearman's rho (Spearman, 1904, 1906), Pearson's correlation coefficient and parametric copulas (Embrechts et al., 2002) are the most widely used dependence measures for quantifying dependence between different financial return series. It is known that Kendall's tau and Spearman's rho can be expressed via copula technology (Schweizer and Wolff, 1981), thus they are also named as copula-based dependence measures. Now, copula theory and rank dependence measures are widely used in a variety of applications, especially in finance (see, Cherubini et al., 2004; McNeil et al., 2005; Patton, 2006, 2012).

It is known that, traditional dependence measures, except for tail dependence coefficients, all analyze dependence between two random variables from a global perspective. For instance, in the field of finance, copula technology is widely used in modeling the global dependence structure between different return series (Rodriguez, 2007; Okimoto, 2008; Chollete et al., 2009; Ning, 2010). However, can global dependence between two variables represent local dependence in the regions that we are interested in? The answer is no. For example, we usually use global Pearson's correlation coefficient to analyze the global correlation between different stock markets, whereas some researches show that the conditional correlation increase in bear market, while it does not seem to increase in bull market, which implies an asymmetric correlation pattern between bear market and bull market (Longin and Solnik, 2001; Ang and Chen, 2002; Campbell et al., 2008; Chung et al. 2019). Therefore, if global dependence measures are applied in studying the dependence between two stock markets, they might cover much useful local dependence information, such as the changing trend of the local dependence as the increase or decrease of returns, and the symmetric or asymmetric dependence patterns between bear market and bull market. These information is useful for helping people deepen their understanding of the relationship between two stock markets. The aim of this paper is to propose a novel copula-based local dependence framework to uncover richer local dependence information between two financial return series.

To our knowledge, the topic of local dependence has already attracted researchers' attention. There are several definitions of local dependence. Some of them are extensions of Pearson's linear correlation. For example, Bjerve and Doksum (1993) and Doksum et al. (1994) study the correlation curve generated by local dependence function on the condition of $X = x$, which reveals the changing tendency of local Pearson's correlation coefficient as the increase or decrease of $X = x$. Longin and Solnik (2001) and Ang and Chen (2002) propose exceedance correlation to investigate the asymmetry characteristics between different equity markets, which can also be regarded as a local version of Pearson's correlation coefficient. Tjøstheim and Hufthammer (2013) propose a measure named local Gaussian correlation, which is based on the estimation of the local Gaussian densities around any selected points (Hjort and Jones 1996). They use this measure to draw local Gaussian correlation map for two continuous variables, which can help people to visualize the changing characteristics of the local dependence in extremely small regions intuitively. In addition, researchers also propose some nonlinear local dependence measures. For example, Mari and Kotz (2001, p.172) and Balakrishnan and Lai (2009, p.169) introduce a local version of Kendall's tau and Spearman's rho, which restrict the calculation region of Kendall's tau and Spearman's rho to a region around certain point. Other nonlinear local dependence measures can be found in Sibuya (1960), Holland and Wang (1987), Oakes (1989), Jones (1996; 1998; 2003) and Li et al. (2014).

Above local dependence measures define local dependence from different perspectives respectively, among which Longin and Solnik's (2001) exceedance correlation is a popular tool for measuring the linear symmetric and asymmetric correlation between different return series (see, for example, Okimoto, 2008; Kang et al., 2010). But the dependence between return series is actually a kind of nonlinear relationship. Therefore, using linear correlation coefficient to measure the dependence between different returns may be misleading because it may not cover the whole range of dependence from -1 to +1 (Cherubini, 2004, p.42). Moreover, it cannot be estimated via copula directly. Kendall's tau is a rank-based dependence measure, which is suitable for measuring nonlinear relationship. Although Manner (2010) introduces the concept of exceedance Kendall's tau and provides its copula-based expressions, these expressions are based on conditional expectation, which cannot be evaluated analytically. In addition, exceedance Kendall's tau is defined along the main diagonal.[1] Such definition, on the one hand, is not appropriate for measuring the negative dependence between return series, on the other hand, definition along the diagonal is too restrictive.

In this paper, we propose a general copula-based local Kendall's tau framework which could uncover richer local dependence information between two return series via their dependence structure (copula) directly. Specifically, this paper makes the following contributions to the existing literature: First, before proposing the copula-based local Kendall's tau framework, we extend exceedance Kendall's tau to the more generalized local Kendall's tau framework. Its basic principle is similar to the exceedance correlation proposed by Longin and Solnik (2001) and the reverse threshold correlation proposed by Christoffersen and Langlois (2013). However, the threshold parameters of exceedance correlation and reverse threshold correlation are all defined on diagonals. We release such restrictions, and extend them to a wider dependence framework whose threshold parameters are not on diagonals anymore. Global dependence and tail dependence become two special cases of this general framework. Second, we propose the copula-based formulas of different local Kendall's tau, which is the most important contribution of our work. Unlike Manner's (2010) copula-based formulas of exceedance Kendall's tau, our formulas of local Kendall's tau are closed form solutions in terms of copula. Using copula function to calculate local Kendall's tau is appealing, because it could link local dependence between two variables with their global dependence structure together,

[1] The concept of main diagonal is introduced in the introduction of Figure 1.

which means that we can directly estimate the local dependence between two variables via the copula function between them. It is well-known that Schweizer and Wolff (1981) proposed the mathematical formula between global Kendall's tau and copula function. Our formulas provide a generalized framework for investigating richer dependence information than Schweizer and Wolff's (1981) formula. Third, based on the copula-based local Kendall's tau framework, we further develop a new class of tail dependence coefficient measuring rank-based dependence in different tails, via which we can study the rank-based tail dependence of a copula function. Fourth, we extend the copula-based local dependence framework to Spearman's rho.

In the empirical section of this article, we illustrate the advantages of copula-based local Kendall's tau relative to global Kendall's tau with stock market data. The results indicate that copula-based local Kendall's tau could uncover richer dependence information than global Kendall's tau, which could deepen our understanding of the dependence between two return series. Specifically, we use copula-based local Kendall's tau and global Kendall's tau respectively to study co-movement between U.S. and other five international stock markets, and get following conclusions: first, global Kendall's tau might overestimate or underestimate the degree of co-movement between two stock markets when both of them are in boom situation or crash situation; second, not only can copula-based local Kendall's tau characterize the symmetric and asymmetric co-movement patterns between two stock markets, but also the "V type" and "inversed V type" co-movement patterns, whereas global Kendall's tau could not.

The rest of the article is organized as follows. In Section 2, we introduce the copula-based local Kendall's tau framework and some of its extensions. In Section 3, we draw and compare theoretical and empirical local Kendall's tau surfaces for some common used Archimedean copulas. In Section 4, we illustrate the advantages of copula-based local Kendall's tau relative to global Kendall's tau with stock market data. Section 5 concludes.

2 The Copula-based Local Dependence Framework

In this section, we first introduce the concept of the general local dependence framework based on Kendall's tau and its non-parametric estimation approach, then we focus on introducing its parametric copula-based estimation approach. Next, as extension of the copula-based local Kendall's tau framework, we develop a new class of tail dependence measure. Finally, we extend the copula-based local dependence framework to Spearman's rho.

2.1 Definition of the general local Kendall's tau framework

It is known that Kendall's tau (Kendall, 1938) is the most widely used rank correlation coefficient. In the past literatures, scholars usually use global Kendall's tau to analyze global dependence between variables. The global Kendall's tau is defined by (see, Nelsen, 2006; Lehmann and D'Abrera, 2006): Let $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ be random samples from the n observations of the vector pair (X, Y) , then we can get C_n^2 distinct pairs (x_i, y_i) and (x_j, y_j) from the global sample. Let c denotes the number of concordant pairs and d denotes the number of discordant pairs, then global Kendall's tau is described by the probability of the concordant pairs minus the probability of the discordant pairs in global sample

$$\tau(X,Y) = \frac{Number\ of\ concordant\ pairs - Number\ of\ discordant\ pairs}{Number\ of\ total\ pairs} = \frac{c-d}{c+d} = \frac{c-d}{C_n^2}$$

Nelsen (2006) introduces the population version of global Kendall's tau for vector (X, Y) with continuous random variables: Let (X_1, Y_1) and (X_2, Y_2) be two independent realizations of a joint distribution, then global Kendall's tau is defined as

$$\tau\left(X,Y\right) = P\left[\left(X_1 - X_2\right)\left(Y_1 - Y_2\right) > 0\right] - P\left[\left(X_1 - X_2\right)\left(Y_1 - Y_2\right) < 0\right] \tag{1}$$

As we know, global Kendall's tau just focuses on global observations. In the following, we propose a general local dependence framework based on Kendall's tau, which consists of four different types of local dependence measure.

Definition 1. (The general Local Kendall's tau framework) Let X and Y be two independent and continuous random variables following different distributions. Local Kendall's tau is defined as

$$\tau_{UU}^{Kendall}(X,Y;p,q) \quad = \tau_{UU}^{Kendall}(X,Y \mid \Omega_{UU} : X \geq F_X^{-1}(p), Y \geq F_Y^{-1}(q)) \tag{2}$$
$$\tau_{UL}^{Kendall}(X,Y;p,q) \quad = \tau_{UL}^{Kendall}(X,Y \mid \Omega_{UL} : X \geq F_X^{-1}(p), Y \leq F_Y^{-1}(q)) \tag{3}$$
$$\tau_{LU}^{Kendall}(X,Y;p,q) \quad = \tau_{LU}^{Kendall}(X,Y \mid \Omega_{LU} : X \leq F_X^{-1}(p), Y \geq F_Y^{-1}(q)) \tag{4}$$
$$\tau_{LL}^{Kendall}(X,Y;p,q) \quad = \tau_{LL}^{Kendall}(X,Y \mid \Omega_{LL} : X \leq F_X^{-1}(p), Y \leq F_Y^{-1}(q)) \tag{5}$$

where $F_X(\cdot)$ and $F_Y(\cdot)$ are marginals with quantiles p and q, $0 \leq p \leq 1$ and $0 \leq q \leq 1$. Ω_{UU}, Ω_{UL}, Ω_{LU} and Ω_{LL} are conditional events for different local Kendall's tau.

From the above local dependence framework, we can see that they are four types of conditional Kendall's tau. Their conditional events restrict the scales of different regions via two quantiles. When $p \to 0$ and $q \to 0$ for Ω_{UU}, $p \to 0$ and $q \to 1$ for Ω_{UL}, $p \to 1$ and $q \to 0$ for Ω_{LU}, $p \to 1$ and $q \to 1$ for Ω_{LL}, above conditional dependence measures become the global dependence measure. When $p \to 1$ and $q \to 1$ for Ω_{UU}, $p \to 1$ and $q \to 0$ for Ω_{UL}, $p \to 0$ and $q \to 1$ for Ω_{LU}, $p \to 0$ and $q \to 0$ for Ω_{LL}, they become four classes of tail dependence measures. Therefore, this local dependence framework is actually a dependence benchmarking, which nests global dependence measure and tail dependence measure. In addition, Manner's (2010) exceedance Kendall's tau is a special case of this framework along the main diagonal. If replacing Kendall's tau by Pearson's linear correlation coefficient, when $p = 1 - q$ for upper-lower and lower-upper local dependence measures, the reverse threshold correlation introduced by Christoffersen and Langlois (2013) also become its special case.

To further illustrate the concept of this framework, we use $U = F_X(X)$ and $V = F_Y(Y)$ to transform the original random variables X and Y to their corresponding i.i.d. variables U and V respectively. We use Clayton copula with parameter 2 ($\theta_C = 2$) to generate 5000 random data,[2] and put them into the two-dimensional coordinate based on U-V panel (see Figure 1). The dark-colored (red) parts in Figure 1 represent regions for calculating upper-upper local Kendall's tau $\tau_{UU}^{Kendall}(U, V; 0.8, 0.8)$, upper-lower local Kendall's tau $\tau_{UL}^{Kendall}(U, V; 0.8, 0.2)$, lower-upper local Kendall's tau $\tau_{LU}^{Kendall}(U, V; 0.2, 0.8)$ and lower-lower local Kendall's tau $\tau_{LL}^{Kendall}(U, V; 0.2, 0.2)$ respectively. The light-colored (yellow) parts represent regions for calculating upper-upper local Kendall's tau $\tau_{UU}^{Kendall}(U, V; 0.4, 0.4)$, upper-lower local Kendall's tau $\tau_{UL}^{Kendall}(U, V; 0.7, 0.3)$, lower-upper local Kendall's tau $\tau_{LU}^{Kendall}(U, V; 0.3, 0.6)$ and lower-lower local Kendall's tau $\tau_{LL}^{Kendall}(U, V; 0.4, 0.3)$ respectively. The whole sample in the coordinate is used for calculating global Kendall's tau. We can see that the 5000 random data generated by bivariate Clayton copula exhibit obvious asymmetric characteristic in upper-upper and lower-lower regions. This local Kendall's tau framework can be used for uncovering whether there exist differences among local dependence in different regions.

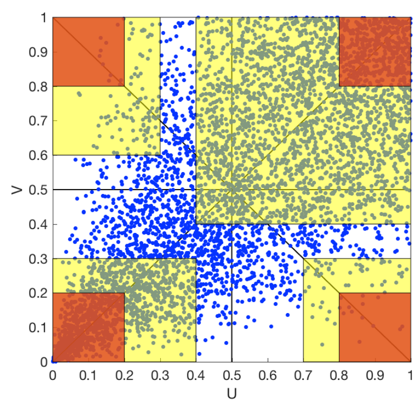

Figure 1. Regions for different local Kendall's tau. The line from (0,0) to (1,1) is the main diagonal. The line from (0,1) to (1,0) is the minor diagonal. The dark-colored (red) parts represent four symmetric regions along the main and minor diagonals, while the light-colored (yellow) parts represent four different asymmetric regions along the main and minor diagonals. These regions are defined by conditional events in expressions (2)(3)(4)(5) respectively.

2.2 Non-parametric estimators of local Kendall's tau framework

Before proposing the copula-based estimation approach of the local Kendall's tau framework, we first introduce the non-parametric estimators of different local Kendall's tau. As we know, the global Kendall's tau is actually the expectation of its kernel function $h((X_1, Y_1), (X_2, Y_2)) = sign(X_1 -$

[2] The expression of Clayton copula can be found in section 3.

$X_2) \cdot sign(Y_1 - Y_2)$: $\tau = E\{sign(X_1 - X_2) \cdot sign(Y_1 - Y_2)\}$. Its consistent estimator is defined by the U-statistic (Kruskal, 1958; Randles and Wolfe, 1991)

$$\widehat{\tau}(X, Y) = \frac{1}{M} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Big\{ sign(X_i - X_j) \cdot sign(Y_i - Y_j) \Big\}$$

The sign function equals +1, 0 and -1 respectively according to whether its argument is positive, zero or negative. n is the number of observations. $M = C_n^2$ is the number of total observation pairs.

In fact, the local Kendall's tau defined in Section 2.1 belongs to a class of conditional Kendall's tau, thus different local Kendall's tau can be estimated as the expectations of their kernel function under different conditional events

$$\begin{aligned}
\tau_{UU}^{Kendall}(X, Y; p, q) &= E\big\{sign(X_1 - X_2) \cdot sign(Y_1 - Y_2) \mid \Omega_{UU}^{12}\big\} \\
&= E\big\{sign(X_1 - X_2) \cdot sign(Y_1 - Y_2) \cdot I(\Omega_{UU}^{12})\big\} / Pr(\Omega_{UU}^{12}) \\
\tau_{UL}^{Kendall}(X, Y; p, q) &= E\big\{sign(X_1 - X_2) \cdot sign(Y_1 - Y_2) \mid \Omega_{UL}^{12}\big\} \\
&= E\big\{sign(X_1 - X_2) \cdot sign(Y_1 - Y_2) \cdot I(\Omega_{UL}^{12})\big\} / Pr(\Omega_{UL}^{12}) \\
\tau_{LU}^{Kendall}(X, Y; p, q) &= E\big\{sign(X_1 - X_2) \cdot sign(Y_1 - Y_2) \mid \Omega_{LU}^{12}\big\} \\
&= E\big\{sign(X_1 - X_2) \cdot sign(Y_1 - Y_2) \cdot I(\Omega_{LU}^{12})\big\} / Pr(\Omega_{LU}^{12}) \\
\tau_{LL}^{Kendall}(X, Y; p, q) &= E\big\{sign(X_1 - X_2) \cdot sign(Y_1 - Y_2) \mid \Omega_{LL}^{12}\big\} \\
&= E\big\{sign(X_1 - X_2) \cdot sign(Y_1 - Y_2) \cdot I(\Omega_{LL}^{12})\big\} / Pr(\Omega_{LL}^{12})
\end{aligned}$$

where (X_k, Y_k), $k = 1, 2$, are observations from the distribution of (X, Y); $\Omega_{UU}^{12} : X_k \geq F_X^{-1}(p) \bigcap Y_k \geq F_Y^{-1}(q)$, $\Omega_{UL}^{12} : X_k \geq F_X^{-1}(p) \bigcap Y_k \leq F_Y^{-1}(q)$, $\Omega_{LU}^{12} : X_k \leq F_X^{-1}(p) \bigcap Y_k \geq F_Y^{-1}(q)$ and $\Omega_{LL}^{12} : X_k \leq F_X^{-1}(p) \bigcap Y_k \leq F_Y^{-1}(q)$, $k = 1, 2$, denote conditional events that two observations are restricted in different regions. $I(A)$ denotes the indicator function with argument A. When A is true, $I(A) = 1$, otherwise, $I(A) = 0$. $Pr(A)$ denotes the probability of event A. We define $Pr(\Omega_{UU}^{12}) = \mu_{UU}$, $Pr(\Omega_{UL}^{12}) = \mu_{UL}$, $Pr(\Omega_{LU}^{12}) = \mu_{LU}$ and $Pr(\Omega_{LL}^{12}) = \mu_{LL}$, respectively.

Following Martin and Betensky (2005), we propose the consistent estimators of $\tau_{UU}^{Kendall}$, $\tau_{UL}^{Kendall}$, $\tau_{LU}^{Kendall}$ and $\tau_{LL}^{Kendall}$ respectively, which are expressed by the ratio of two U-statistics

$$\begin{aligned}
\widehat{\tau}_{UU}^{Kendall}(X, Y; p, q) &= \frac{1}{M_{UU}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Big\{ sign(X_i - X_j) \cdot sign(Y_i - Y_j) \cdot I(\Omega_{UU}^{ij}) \Big\} \\
&= \frac{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \{sign(X_i - X_j) \cdot sign(Y_i - Y_j) \cdot I(\Omega_{UU}^{ij})\}}{C_n^2} \Big/ \left(\frac{M_{UU}}{C_n^2} \right) = \frac{U_{UU}}{U_{UU}^M} \\
\widehat{\tau}_{UL}^{Kendall}(X, Y; p, q) &= \frac{1}{M_{UL}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Big\{ sign(X_i - X_j) \cdot sign(Y_i - Y_j) \cdot I(\Omega_{UL}^{ij}) \Big\}
\end{aligned}$$

$$= \frac{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{sign(X_i - X_j)\cdot sign(Y_i - Y_j)\cdot I(\Omega_{UL}^{ij})\}}{C_n^2} \Big/ \left(\frac{M_{UL}}{C_n^2}\right) = \frac{U_{UL}}{U_{UL}^M}$$

$$\widehat{\tau}_{LU}^{Kendall}(X,Y;p,q) = \frac{1}{M_{LU}}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\Big\{sign(X_i - X_j)\cdot sign(Y_i - Y_j)\cdot I(\Omega_{LU}^{ij})\Big\}$$

$$= \frac{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{sign(X_i - X_j)\cdot sign(Y_i - Y_j)\cdot I(\Omega_{LU}^{ij})\}}{C_n^2} \Big/ \left(\frac{M_{LU}}{C_n^2}\right) = \frac{U_{LU}}{U_{LU}^M}$$

$$\widehat{\tau}_{LL}^{Kendall}(X,Y;p,q) = \frac{1}{M_{LL}}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\Big\{sign(X_i - X_j)\cdot sign(Y_i - Y_j)\cdot I(\Omega_{LL}^{ij})\Big\}$$

$$= \frac{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\{sign(X_i - X_j)\cdot sign(Y_i - Y_j)\cdot I(\Omega_{LL}^{ij})\}}{C_n^2} \Big/ \left(\frac{M_{LL}}{C_n^2}\right) = \frac{U_{LL}}{U_{LL}^M}$$

where $\Omega_{UU}^{ij}: X_k \geq F_X^{-1}(p)\bigcap Y_k \geq F_Y^{-1}(q)$, $\Omega_{UL}^{ij}: X_k \geq F_X^{-1}(p)\bigcap Y_k \leq F_Y^{-1}(q)$, $\Omega_{LU}^{ij}: X_k \leq F_X^{-1}(p)$ $\bigcap Y_k \geq F_Y^{-1}(q)$ and $\Omega_{LL}^{ij}: X_k \leq F_X^{-1}(p)\bigcap Y_k \leq F_Y^{-1}(q)$, $k = i, j$, denote conditional events that two observations are restricted in different regions. M_{UU}, M_{UL}, M_{LU} and M_{LL} represent the number of observation pairs in different regions, which are calculated by $M_{UU} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} I(\Omega_{UU}^{ij})$, $M_{UL} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} I(\Omega_{UL}^{ij})$, $M_{LU} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} I(\Omega_{LU}^{ij})$ and $M_{LL} = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} I(\Omega_{LL}^{ij})$ respectively. U_{UU}^M, U_{UL}^M, U_{LU}^M and U_{LL}^M are U-statistics with expected values μ_{UU}, μ_{UL}, μ_{LU} and μ_{LL} respectively. U_{UU}, U_{UL}, U_{LU} and U_{LL} are another kind of U-statistics with expected values $\tau_{UU}^{Kendall}\mu_{UU}$, $\tau_{UL}^{Kendall}\mu_{UL}$, $\tau_{LU}^{Kendall}\mu_{LU}$ and $\tau_{LL}^{Kendall}\mu_{LL}$.

Applying the one-sample U-statistic theorem (Randles and Wolfe 1991), $\sqrt{n}(\widehat{\tau}_{UU}^{Kendall} - \tau_{UU}^{Kendall})$, $\sqrt{n}(\widehat{\tau}_{UL}^{Kendall} - \tau_{UL}^{Kendall})$, $\sqrt{n}(\widehat{\tau}_{LU}^{Kendall} - \tau_{LU}^{Kendall})$ and $\sqrt{n}(\widehat{\tau}_{LL}^{Kendall} - \tau_{LL}^{Kendall})$ are asymptotically $N(0, 4\zeta_{UU}/\mu_{UU}^2)$, $N(0, 4\zeta_{UL}/\mu_{UL}^2)$, $N(0, 4\zeta_{LU}/\mu_{LU}^2)$, $N(0, 4\zeta_{LL}/\mu_{LL}^2)$ provided that

$$\begin{aligned}
\zeta_{UU} = E\{&sign(X_1 - X_2)\cdot sign(Y_1 - Y_2) \\
&\times sign(X_1 - X_3)\cdot sign(Y_1 - Y_3)\cdot I(\Omega_{UU}^{12}\wedge\Omega_{UU}^{13})\} - (\tau_{UU}^{Kendall}\mu_{UU})^2 \\
\zeta_{UL} = E\{&sign(X_1 - X_2)\cdot sign(Y_1 - Y_2) \\
&\times sign(X_1 - X_3)\cdot sign(Y_1 - Y_3)\cdot I(\Omega_{UL}^{12}\wedge\Omega_{UL}^{13})\} - (\tau_{UL}^{Kendall}\mu_{UL})^2 \\
\zeta_{LU} = E\{&sign(X_1 - X_2)\cdot sign(Y_1 - Y_2) \\
&\times sign(X_1 - X_3)\cdot sign(Y_1 - Y_3)\cdot I(\Omega_{LU}^{12}\wedge\Omega_{LU}^{13})\} - (\tau_{LU}^{Kendall}\mu_{LU})^2 \\
\zeta_{LL} = E\{&sign(X_1 - X_2)\cdot sign(Y_1 - Y_2) \\
&\times sign(X_1 - X_3)\cdot sign(Y_1 - Y_3)\cdot I(\Omega_{LL}^{12}\wedge\Omega_{LL}^{13})\} - (\tau_{LL}^{Kendall}\mu_{LL})^2
\end{aligned}$$

are all positive. Note that X_1, X_2 and X_3 are i.i.d.

According to Martin and Betensky (2005), the consistent estimators of ζ_{UU}, ζ_{UL}, ζ_{LU} and ζ_{LL} can be obtained by $\widehat{\zeta}_{UU} = (2n\cdot C_{n-1}^2)^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\sum_{k\neq i,j}\big\{sign(X_i - X_j)\cdot sign(Y_i - Y_j)\cdot I(\Omega_{UU}^{ij})\cdot sign(X_i - X_k)\cdot sign(Y_i - Y_k)\cdot I(\Omega_{UU}^{ik})\big\}$, $\widehat{\zeta}_{UL} = (2n\cdot C_{n-1}^2)^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\sum_{k\neq i,j}\big\{sign(X_i - X_j)\cdot sign(Y_i - Y_j)\cdot I(\Omega_{UL}^{ij})\cdot sign(X_i - X_k)\cdot sign(Y_i - Y_k)\cdot I(\Omega_{UL}^{ik})\big\}$, $\widehat{\zeta}_{LU} = (2n\cdot C_{n-1}^2)^{-1}\sum_{i=1}^{n}\sum_{j\neq i}\sum_{k\neq i,j}$

$\{sign(X_i - X_j) \cdot sign(Y_i - Y_j) \cdot I(\Omega_{LU}^{ij}) \cdot sign(X_i - X_k) \cdot sign(Y_i - Y_k) \cdot I(\Omega_{LU}^{ik})\}$ and $\widehat{\zeta}_{LL} = (2n \cdot C_{n-1}^2)^{-1} \sum_{i=1}^n \sum_{j \neq i} \sum_{k \neq i,j} \{sign(X_i - X_j) \cdot sign(Y_i - Y_j) \cdot I(\Omega_{LL}^{ij}) \cdot sign(X_i - X_k) \cdot sign(Y_i - Y_k) \cdot I(\Omega_{LL}^{ik})\}$ respectively.

2.3 Copula-based estimation approach

Although the non-parametric estimation method has its advantages, e.g., avoiding any kind of model misspecification, the parametric estimation method is convenient to implement in the empirical study. In the following, we will focus on introducing the parametric copula-based estimation approach of the local Kendall's tau framework.

Copula is the function that can join different variables following various distributions, and thus can be used to characterize the dependence structure between different variables. According to Sklar's (1959) Theorem, a bivariate joint cumulative function of two variables can be decomposed into three components: two marginal cumulative distribution functions and their dependence structure. Specifically, let $F(X,Y)$, $F_X(X)$ and $F_Y(Y)$ represent the joint distribution, the marginal distribution of variable X, and the marginal distribution of variable Y respectively. Then, $F(X,Y)$ can be expressed via copula function C and the marginal distributions of X and Y

$$F(X,Y) = C(F_X(X), F_Y(Y)) = C(U,V)$$

where $U = F_X(X)$, $V = F_Y(Y)$. Its major advantage is that it allows us to separate the dependence structure from the marginal distributions, and captures the properties of joint distribution that are invariant under strictly increasing transformations (Embrechts et al. 2002).

Schweizer and Wolff (1981) introduced the copula-based formula of global Kendall's tau, thus copula technology also inspires us to propose a copula-based method to estimate local Kendall's tau. First, let us recall Schweizer and Wolff's (1981) formula of global Kendall's tau: Let C denotes the copula of (X,Y), then global Kendall's tau of X and Y can be calculated by following formula

$$\tau(X,Y) = 4\int_0^1 \int_0^1 C(u,v)dC(u,v) - 1 \tag{6}$$

We can see from above expression that global Kendall's tau between X and Y is determined by their dependence structure. Via this formula, we can estimate global Kendall's tau from parametric copula directly (Cherubini et al. 2004, p. 126).

As extension of Schweizer and Wolff's (1981) work, we derive the closed form solutions of different local Kendall's tau in terms of copula, which enable us to investigate the rank-based local dependence between two variables via their global dependence structure directly.

Theorem 1. Let (X,Y) be a pair of continuous random variables following different distributions $F_X(\cdot)$ and $F_Y(\cdot)$ respectively, p and q are their quantiles. C denotes the copula of (X,Y). Different local Kendall's tau can be expressed by

$$\tau_{UU}^{Kendall}(X,Y;p,q) = \frac{2\int_q^1 \int_p^1 [2C(u,v) - C(p,v) - C(u,q) - u - v]dC(u,v)}{(1-p-q+C(p,q))^2} + \frac{1+p+q+C(p,q)}{1-p-q+C(p,q)} \quad (7)$$

$$\tau_{UL}^{Kendall}(X,Y;p,q) = \frac{2\int_0^q \int_p^1 [2C(u,v) - C(p,v) - v]dC(u,v)}{(q-C(p,q))^2} \quad (8)$$

$$\tau_{LU}^{Kendall}(X,Y;p,q) = \frac{2\int_q^1 \int_0^p [2C(u,v) - C(u,q) - u]dC(u,v)}{(p-C(p,q))^2} \quad (9)$$

$$\tau_{LL}^{Kendall}(X,Y;p,q) = \frac{4\int_0^q \int_0^p C(u,v)dC(u,v)}{C(p,q)^2} - 1 \quad (10)$$

Proof: See Appendix A.

Corollary 1. When $p \to 0$ and $q \to 0$, the copula-based formula of upper-upper local Kendall's tau reduce to copula-based formula of global Kendall's tau

$$\lim_{p\to 0}\lim_{q\to 0} \tau_{UU}^{Kendall}(X,Y;p,q) = 4\int_0^1 \int_0^1 C(u,v)dC(u,v) - 1$$

Proof: See Appendix B.

Similarly, the formulas of the other three local Kendall's tau can also reduce to the formula of global Kendall's tau, which indicates that our formulas for local Kendall's tau are different generalized versions of formula (6).

Let $p = q = z$, $0 < z < 1$, the lower-lower local Kendall's tau will reduce to the formula of the so-called cumulative tau that is introduced by Venter (2002)

$$\lim_{p\to z}\lim_{q\to z} \tau_{LL}^{Kendall}(X,Y;p,q) = \lim_{p\to z}\lim_{q\to z} \frac{4\int_0^p \int_0^q C(u,v)dC(u,v)}{C(p,q)^2} - 1 = \frac{4\int_0^z \int_0^z C(u,v)dC(u,v)}{C(z,z)^2} - 1$$

Therefore, Venter's (2002) cumulative tau is also a special case of this framework.

From formulas (7)(8)(9)(10), we can infer that local Kendall's tau between two random variables X and Y is actually the function of threshold parameters p, q and their underlying copula. In other words, once the copula between X and Y is chosen, their local dependence is determined by the threshold parameters p and q. These formulas uncover the relationship between copula function and Kendall's tau in different local regions, providing richer dependence information than global dependence measure, and therefore we could directly use them to study the local dependence between two variables via their global dependence structure.

2.4 Relationship between different local Kendall's tau

The local Kendall's tau framework contains four different definitions. In fact, there exist certain relationship among them. For example, $\tau_{LL}^{Kendall}$ is given by

$$\tau_{LL}^{Kendall}(X, Y; p, q) = \tau_{LL}^{Kendall}(X, Y \mid \Omega_{LL} : X \leq F_X^{-1}(p), Y \leq F_Y^{-1}(q))$$

Then, $\tau_{UL}^{Kendall}$, $\tau_{LU}^{Kendall}$ and $\tau_{UU}^{Kendall}$ can be computed by following transformations

$$\tau_{UL}^{Kendall}(X, Y; p, q) = -\tau_{LL}^{Kendall}(-X, Y; 1-p, q) \tag{11}$$

$$\tau_{LU}^{Kendall}(X, Y; p, q) = -\tau_{LL}^{Kendall}(X, -Y; p, 1-q) \tag{12}$$

$$\tau_{UU}^{Kendall}(X, Y; p, q) = \tau_{LL}^{Kendall}(-X, -Y; 1-p, 1-q) \tag{13}$$

Proof: See Appendix C.

2.5 Extension: Tail dependence measures based on local Kendall's tau

It is known that the tail dependence coefficient proposed by Ledford and Tawn (1996; 1997) is a famous measure characterizing extreme dependence between two random variables. From their formulas shown in (14) and (15), we can see that the main principle of this measure is calculating the extreme values of conditional probability between two variables. Copula-based formulas of λ_{UU} and λ_{LL} are given by

$$\lambda_{UU} = \lim_{p \to 1^-} Pr\{X > F_X^{-1}(p) \mid Y > F_Y^{-1}(p)\} = \lim_{p \to 1^-} \frac{1 - 2p + C(p,p)}{1-p} \tag{14}$$

$$\lambda_{LL} = \lim_{p \to 0^+} Pr\{X \leq F_X^{-1}(p) \mid Y \leq F_Y^{-1}(p)\} = \lim_{p \to 0^+} \frac{C(p,p)}{p} \tag{15}$$

where p represents the quantile of X and Y. λ_{UU} and λ_{LL} are upper tail dependence coefficient and lower tail dependence coefficient bounded between 0 and 1. If their values tend to 1, X and Y exhibit a strong tail dependence, while if their values tend to 0, two variables exhibit weak tail dependence. Zhang (2008) extended λ_{UU} and λ_{LL} to total tail dependence along both the main diagonal and the minor diagonal, which could capture richer dependence information.

Similarly, we study the limits of local Kendall's tau when two variables tend to their extreme values respectively, and construct a novel class of tail dependence measures based on local Kendall's tau. In what follow, we will introduce its definition and copula-based estimation method.

Definition 3. (Tail dependence measures based on Local Kendall's tau) X and Y are two random variables, p and q are their quantiles respectively. The extreme values of different local Kendall's tau represent the degree of rank-based tail dependence between two variables, which are defined by

$$\lambda_{UU}^{Kendall} = \lim_{p \to 1} \tau_{UU}^{Kendall}(X, Y; p, p) \tag{16}$$

$$\lambda_{UL}^{Kendall} = \lim_{p \to 1} \tau_{UL}^{Kendall}(X, Y; p, 1-p) \tag{17}$$

$$\lambda_{LU}^{Kendall} = \lim_{p \to 0} \tau_{LU}^{Kendall}(X, Y; p, 1-p) \tag{18}$$

$$\lambda_{LL}^{Kendall} = \lim_{p \to 0} \tau_{LL}^{Kendall}(X, Y; p, p) \tag{19}$$

where $\lambda_{UU}^{Kendall}$, $\lambda_{UL}^{Kendall}$, $\lambda_{LU}^{Kendall}$ and $\lambda_{LL}^{Kendall}$ represent upper-upper tail dependence measure, upper-lower tail dependence measure, lower-upper tail dependence measure and lower-lower tail dependence measure based on local Kendall's tau respectively.

We note that, one tail dependence measure in the local dependence framework, $\lambda_{LL}^{Kendall}$, coincides with $\theta^{\tau} = lim_{u \to 0} E\{sign(X_1 - X_2) \cdot sign(Y_1 - Y_2) \mid max(X_1, X_2, Y_1, Y_2) \leq u\}$ proposed by Asimit et al.(2016), which was used to detect the presence of asymptotic independence/dependence. However, they just studied whether the value of θ^{τ} for different copulas, e.g., Gumbel copula, Student's t copula and Elliptical copula, is positive or not. Since we have introduced the closed form formulas of different local Kendall's tau in terms of copula in Theorem 1, we can now use copula function to calculate the values of $\lambda_{UU}^{Kendall}$, $\lambda_{UL}^{Kendall}$, $\lambda_{LU}^{Kendall}$ and $\lambda_{LL}^{Kendall}$ via formulas (7)(8)(9)(10) directly.

An example for FGM copula is given below.

Example 1. Tail dependence for Farlie-Gumbel-Morgenstern copula

Farlie-Gumbel-Morgenstern copula (often abbreviated as "FGM") is defined as follows

$$C^{FGM}(u, v, \alpha) = uv + \alpha uv(1-u)(1-v) \quad \alpha \in [-1, 1] \;\; and \;\; (u, v) \in [0, 1]$$

After substituting FGM copula into formulas (7)(8)(9)(10) and (16)(17)(18)(19), we calculate the tail dependence based on Kendall's tau of FGM copula [3]

$$\lambda_{UU}^{Kendall} = \lim_{p \to 1} \tau_{UU}^{Kendall}(X, Y; p, p) = \lim_{p \to 1} \frac{2\alpha(p-1)^2}{9(1+\alpha p^2)^2} = 0$$

$$\lambda_{UL}^{Kendall} = \lim_{p \to 1} \tau_{UL}^{Kendall}(X, Y; p, 1-p) = \lim_{p \to 1} \frac{2\alpha(1-p)^2}{9(1-\alpha p^2)^2} = 0$$

$$\lambda_{LU}^{Kendall} = \lim_{p \to 0} \tau_{LU}^{Kendall}(X, Y; p, 1-p) = \lim_{p \to 0} \frac{2\alpha p^2}{9(1-\alpha(1-p)^2)^2} = 0$$

$$\lambda_{LL}^{Kendall} = \lim_{p \to 0} \tau_{LL}^{Kendall}(X, Y; p, p) = \lim_{p \to 0} \frac{2\alpha p^2}{9(1+\alpha(p-1)^2)^2} = 0$$

From above results, we know that two variables with FGM copula dependence structure tend

[3] The detailed computation procedure of the tail dependence based on local Kendall's tau for FGM copula is available from the authors upon request.

to be independence when they tend to extreme values respectively.

Similarly, the tail dependence based on Kendall's tau of other copulas can also be estimated by formulas (7)(8)(9)(10) and (16)(17)(18)(19) (if their limits exist). Note that, since the former measures are bounded between -1 and 1, whereas λ_{UU} and λ_{LL} are bounded between 0 and 1, the tail dependence measures based on local Kendall's tau are not alternatives to tail dependence coefficients proposed by Ledford and Tawn (1996; 1997).

2.6 Extension to bivariate local Spearman's rho framework

Besides Kendall's tau, there is another famous rank correlation coefficient named rho (Spearman 1904, 1906). Kruskal (1958) introduced the population version of global Spearman's rho: Let (X_1, Y_1), (X_2, Y_2) and (X_3, Y_3) be three independent realizations of a joint distribution, global Spearman's rho is defined by

$$\rho(X,Y) = 6P\left[(X_1 - X_2)(Y_1 - Y_3) > 0\right] - 3 \tag{20}$$

Note that the vectors (X_2, Y_3) can also be replaced by (X_3, Y_2).

Previous researches related with conditional Spearman's rho, e.g., Schmid and Schmidt (2007) and Dobric et al. (2013), all define bivariate conditional Spearman's rho along the main diagonal. Like exceedance Kendall's tau, on the one hand, these definitions are not appropriate for measuring negative dependence between two return series, on the other hand, definitions along the main diagonal are also too restrictive. Therefore, following Definition 1, we also extend the general local dependence framework to bivariate Spearman's rho, and then derive its copula-based formulas.

Definition 2. (The general Local Spearman's rho framework) Let X and Y be two independent and continuous random variables following different distributions. Local Spearman's rho can be expressed by

$$\begin{aligned}
\rho_{UU}^{Spearman}(X,Y;p,q) &= \rho_{UU}^{Spearman}(X,Y \mid \Omega_{UU} : X \geq F_X^{-1}(p), Y \geq F_Y^{-1}(q)) && (21)\\
\rho_{UL}^{Spearman}(X,Y;p,q) &= \rho_{UL}^{Spearman}(X,Y \mid \Omega_{UL} : X \geq F_X^{-1}(p), Y \leq F_Y^{-1}(q)) && (22)\\
\rho_{LU}^{Spearman}(X,Y;p,q) &= \rho_{LU}^{Spearman}(X,Y \mid \Omega_{LU} : X \leq F_X^{-1}(p), Y \geq F_Y^{-1}(q)) && (23)\\
\rho_{LL}^{Spearman}(X,Y;p,q) &= \rho_{LL}^{Spearman}(X,Y \mid \Omega_{LL} : X \leq F_X^{-1}(p), Y \leq F_Y^{-1}(q)) && (24)
\end{aligned}$$

where $F_X(\cdot)$ and $F_Y(\cdot)$ are their marginals with thresholds p and q respectively. $0 \leq p \leq 1$ and $0 \leq q \leq 1$. Ω_{UU}, Ω_{UL}, Ω_{LU} and Ω_{LL} are conditional events for different local Spearman's rho.

Theorem 2. Let (X, Y) be a pair of random variables following different distributions $F_X(\cdot)$ and $F_Y(\cdot)$ respectively, p and q are their quantiles. C denotes the copula of (X, Y). Local Spearman's rho can be expressed by

$$\rho_{UU}^{Spearman}(X,Y;p,q) = \frac{6\int_q^1\int_p^1 \big[2uv-(p+1)v-(q+1)u+pq+1\big]dC(u,v)}{(1-p-q+C(p,q))(1-p-q+pq)} - 3 \tag{25}$$

$$\rho_{UL}^{Spearman}(X,Y;p,q) = \frac{6\int_0^q\int_p^1 \big[2uv-qu-(p+1)v+q\big]dC(u,v)}{q(q-C(p,q))(1-p)} - 3 \tag{26}$$

$$\rho_{LU}^{Spearman}(X,Y;p,q) = \frac{6\int_q^1\int_0^p \big[2uv-(q+1)u-pv+p\big]dC(u,v)}{p(p-C(p,q))(1-q)} - 3 \tag{27}$$

$$\rho_{LL}^{Spearman}(X,Y;p,q) = \frac{6\int_0^q\int_0^p \big[2uv-qu-pv\big]dC(u,v)}{pqC(p,q)} + 3 \tag{28}$$

Proof: See Appendix D.

According to Schweizer and Wolff (1981), the global Spearman's rho can be expressed by

$$\rho(X,Y) = 12\int_0^1\int_0^1 uv dC(u,v) - 3 \tag{29}$$

It is not difficult to prove that formula (29) is a special case of formulas (25)(26)(27)(28).

Schmid and Schmidt (2007) proposed the copula-based formula of multivariate conditional Spearman's rho. In bivariate case, $\rho_{LL}^{Spearman}(X,Y;p,p)$ is expressed by

$$\rho_{LL}^{Spearman}(X,Y;p,p) = \frac{\int_0^p\int_0^p C(u,v)dudv - \frac{p^4}{4}}{\frac{p^3}{3} - \frac{p^4}{4}} \tag{30}$$

There exists some differences between our copula-based formulas of local Spearman's rho and Schmid and Schmidt's (2007) formula. Specifically, formula (30) focuses on the lower-lower case and just has one parameter p, meaning that it can just measure lower-lower local dependence along the main diagonal. But our formulas (25)(26)(27)(28) measure local dependence from four different perspectives, and have two parameters p and q, which make the region of local Spearman's rho to be more flexible.

3 Local dependence surfaces based on Kendall's tau for some bivariate Archimedean copulas

To demonstrate the usefulness of formulas (7)(8)(9)(10) in estimating the theoretical local dependence for a copula function, we use them to draw the theoretical local dependence surfaces for some common used bivariate Archimedean copulas, e.g., Clayton copula, Gumbel copula and Frank copula.[4] The calculation procedure is conducted by Matlab.[5] In fact, we also try Gaussian copula and Student's t copula, but we find that it is hard to calculate their local Kendall's tau via formulas (7)(8)(9)(10) directly.[6] Therefore, we just use some common used Archimedean copulas for example. For comparison, we also draw empirical local dependence surfaces for data simulated by these copulas. Specifically, we first use the selected copulas to generate 50000 random samples respectively.[7] Then we compute the empirical local Kendall's tau of the generated data in different quadrants. Finally, we draw their corresponding empirical local dependence surfaces. To ensure we have enough samples when the regions for calculating the empirical local Kendall's tau are very small, we restrict the quantiles of X and Y in the interval [0.05,0.95].

Clayton copula is a kind of asymmetric copula, which could just capture lower tail dependence and assumes that the upper tail dependence is zero. Thus, this copula has a L-shaped dependence structure. It is given by $C_{Clayton}(u, v; \theta_C) = \left[max(u^{-\theta_C} + v^{-\theta_C} - 1, 0)\right]^{-1/\theta_C}$, where $\theta_C \in [-1, 0]\backslash\{0\}$. Figure 2 shows four local dependence surfaces in different quadrants based on local Kendall's tau for Clayton copula with parameter 1.3 ($\theta_C = 1.3$). We draw its theoretical and empirical local dependence surfaces respectively, and find that the theoretical local dependence surfaces fit well with their corresponding empirical local dependence surfaces in general. The first and the fourth plots indicate that the local dependence surfaces for data generated by Clayton copula exhibit significant asymmetric characteristic along the main diagonal. Specifically, the lower-lower local Kendall's tau keeps at a relatively high level around 0.4, while the upper-upper local Kendall's tau keeps at a low level (less than 0.1). Moreover, we note that the theoretical upper-upper local dependence exhibits decreasing tendency when X and Y tend to upper extreme values. The second and the third plots indicate that lower-upper local dependence surface and upper-lower local dependence surface based on local Kendall's tau for Clayton copula exhibit symmetric characteristic along the minor diagonal, and both of them keep at a low level. We use the mathematical formula between global Kendall's tau and Clayton copula $\tau = \theta_C/(\theta_C+2)$ (Chollete et al., 2011) to calculate its global Kendall's tau: $\tau = 0.3939$. From Figure 2, we observe that the global Kendall's tau for Clayton copula with parameter $\theta_C = 1.3$ is obviously larger than its upper-upper, upper-lower and lower-upper local Kendall's tau. Finally, we note that the upper-lower and lower-upper empirical local dependence surfaces for Clayton copula fluctuate more dramatically when X and Y tend to extreme values along the minor diagonal. It is mainly because the sample size in the upper-lower and lower-upper regions become smaller. If we use rotated Clayton copula (90 degrees), there will be much more data for calculating local Kendall's tau in upper-lower region and lower-upper region, and the fluctuation of the empirical local dependence surfaces will be smaller. In addition,

[4] We also study local dependence surfaces for bivariate FGM copula. The results are available from the authors upon request.

[5] The Matlab codes used in this section are available from the authors upon request.

[6] Finding the simpler closed form solutions of local Kendall's tau for Gaussian and Student's t copulas is the direction of our future research.

[7] We use the internal function provided by Matlab to generate random samples directly.

we could increase the amount of the simulation data to reduce the fluctuation of the empirical local dependence surfaces.

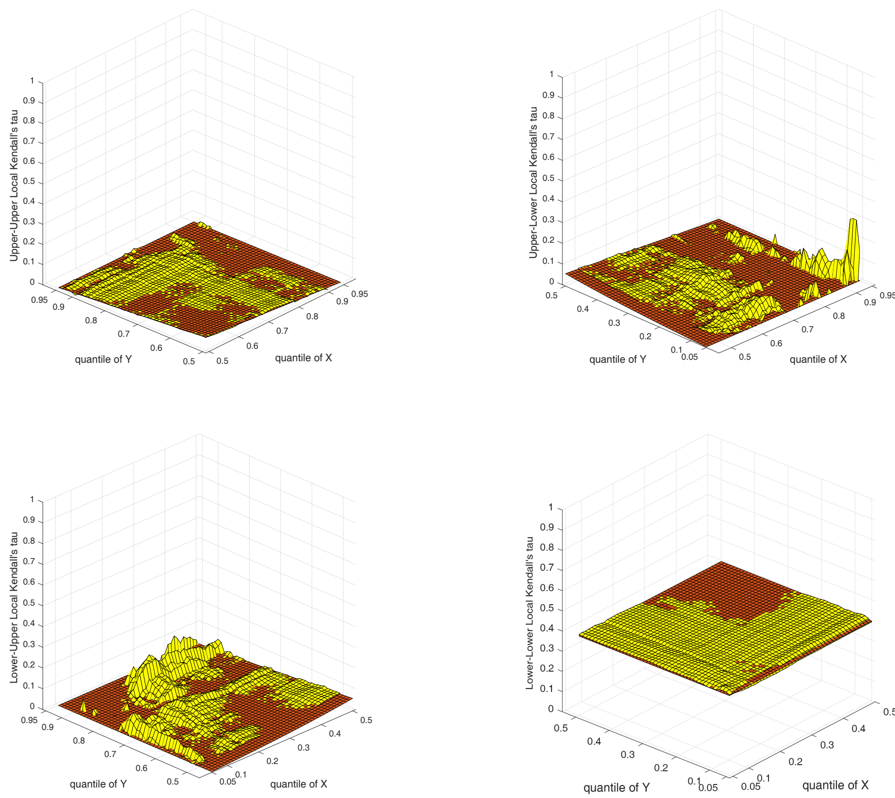

Figure 2. Local dependence surfaces in different quadrants based on Kendall's tau for Clayton copula. Note that the smooth curved surfaces in above plots represent theoretical local dependence surfaces based on local Kendall's tau. The non-smooth surfaces represent empirical local dependence surfaces based on local Kendall's tau.

Gumbel copula is also an asymmetric copula, which could just capture upper tail dependence and assumes that the lower tail dependence is zero. Hence, Gumbel copula has an J-shaped dependence structure. It is given by $C_{Gumbel}(u, v; \delta_G) = \exp\big(-\big[(-log(u))^{\delta_G} + (-log(v))^{\delta_G}\big]^{1/\delta_G}\big)$, where $\delta_G \in [1, +\infty]$. Figure 3 shows the local dependence surfaces in different quadrants based on local Kendall's tau for Gumbel copula with parameter 2 ($\delta_G = 2$). Similar to Clayton copula, we also draw different theoretical and empirical local dependence surfaces for Gumbel copula. We can see that, the theoretical local dependence surfaces also fit well with their corresponding empirical local dependence surfaces. The first and the fourth plots indicate that local Kendall's tau based on Gumbel copula also show significant asymmetric characteristic along the main diagonal. However, the upper-upper local Kendall's tau of Gumbel copula is larger than its lower-lower local Kendall's tau, which is opposite to Clayton copula. We note that the upper-upper local dependence surface based on Kendall's tau for Gumbel copula keeps at a relatively high level around 0.4, whereas the lower-lower local dependence surface keeps at a relatively low level. Similar to Clayton copula, the second and the third plots indicate that the lower-upper local Kendall's tau and upper-lower local Kendall's tau based on Gumbel copula also exhibit symmetric characteristic. Besides the characteristics mentioned above, we also note that lower-lower local Kendall's tau of Gumbel copula shows decreasing tendency as the decrease of two variables, which means the lower-lower local dependence of Gumbel copula becomes weaker when both X and Y tend to lower extreme values. Finally, using the mathematical formula between global Kendall's tau and Gumbel copula $\tau = 1 - 1/\delta_G$ (Genest and Rivest, 1993), we calculate its global Kendall's tau: $\tau = 0.5$. From Figure 3, we also note that the global Kendall's tau for Gumbel copula with parameter $\delta_G = 2$ is obviously larger than its local Kendall's tau in different quadrants. The reason why the upper-lower and lower-upper empirical local Kendall's tau surfaces of Gumbel copula also fluctuate more dramatically when X and Y tend to extreme values along the minor diagonal is the same as Clayton copula.

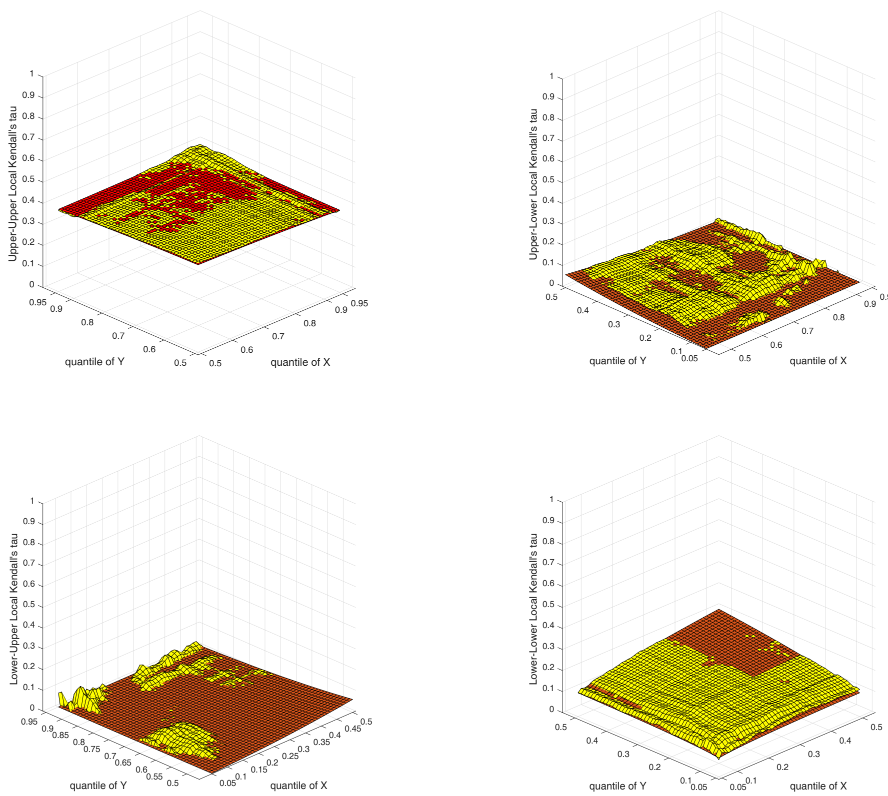

Figure 3. Local dependence surfaces in different quadrants based on Kendall's tau for Gumbel copula. The smooth curved surfaces and non-smooth surfaces represent theoretical and empirical local dependence surfaces based on local Kendall's tau respectively.

Frank copula is symmetric both along the main diagonal and the minor diagonal. It is defined by $C_{Frank}(u, v; \lambda) = -\dfrac{1}{\lambda} ln\left(1 + \dfrac{(e^{-\lambda u} - 1)(e^{-\lambda v} - 1)}{e^{-\lambda} - 1}\right)$, where $\lambda \in \mathbb{R}$, and $\lambda \neq 0$. Figure 4 shows four local Kendall's tau surfaces in four quadrants for Frank copula with parameter 4 ($\lambda = 4$). Its local Kendall's tau surfaces exhibit symmetric characteristic both along the main diagonal and the minor diagonal. Moreover, four local dependence surfaces all show decreasing tendency when two variables tend to extreme values.

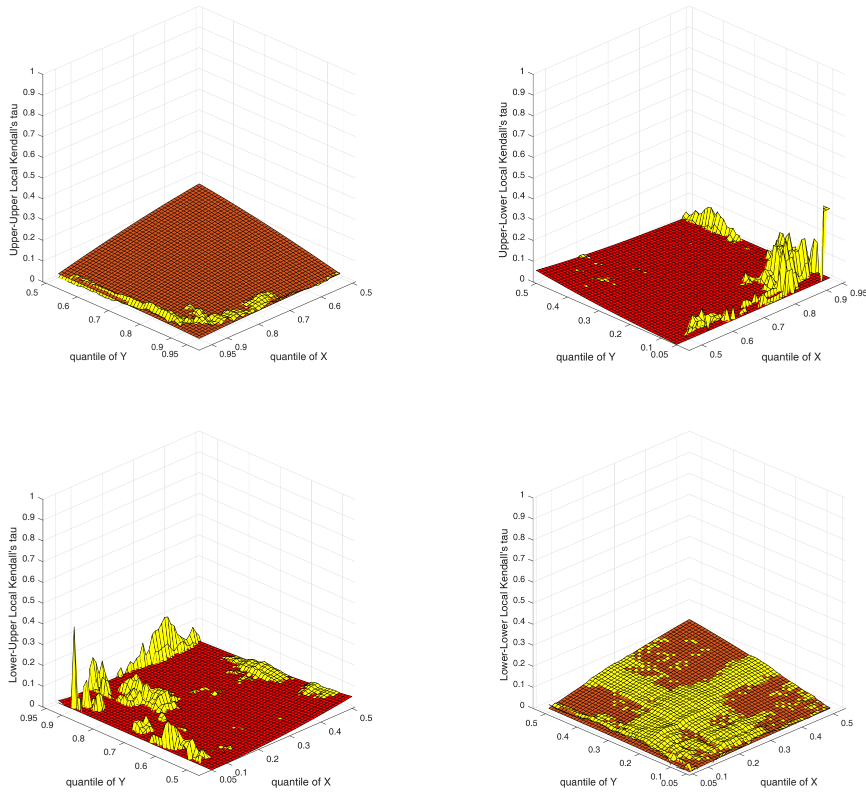

Figure 4. Local dependence surfaces in different quadrants based on Kendall's tau for Frank copula. The smooth curved surfaces and non-smooth surfaces represent theoretical and empirical local dependence surfaces based on local Kendall's tau respectively.

From above examples, we can see that, the copula-based formulas (7)(8)(9)(10) are useful in estimating the theoretical local Kendall's tau of a copula function. Although we just use Clayton copula, Gumbel copula and Frank copula as examples, it is not difficult to infer that theoretical local Kendall's tau of the rotated version or a linear combination of these copulas can also be estimated by formulas (7)(8)(9)(10) directly.

4 An Empirical Study

In this part, we demonstrate the advantages of copula-based local Kendall's tau relative to global Kendall's tau with stock market data. Specifically, we apply copula-based local and global Kendall's tau to study the co-movement relationship between U.S. and other international stock markets respectively, and then compare their performance. Co-movement between stock markets is a kind of nonlinear phenomenon. Relative to the linear correlation coefficient, Kendall's tau is more appropriate for being used in such situation (Li 2014). Since two stock markets usually exhibit positive co-movement relationship, we mainly use two special cases of the local Kendall's tau framework, upper-upper and lower-lower local Kendall's tau along the main diagonal, as examples to study the co-movement between two stock markets under bull market/bull market and bear market/bear market statuses, respectively.

The data consist of daily closing prices p_t for six stock markets from North America, Europe and Asia, including the Standard & Poor's 500 Index (S&P500) in the U.S., the S&P/TSX Composite Index (S&P/TSX) in Canada, the Financial Times Stock Exchange 100 Index (FTSE100) in the U.K., the Hang Seng Index (HSI) in Hong Kong, the Nikkei 225 Stock Average Index (N225) in Japan, and the Shanghai Composite Index (SH) in China. The data are obtained from Yahoo finance over the period January 5, 1998 to July 31, 2017. After deleting all holidays and invalid data, each series consists of 4252 observations. The daily stock return is calculated by multiplying the first difference of the natural logarithm of close price by 100: $R_t = (ln(p_t) - ln(p_{t-1})) \times 100$. As we know, other stock markets, except for Canadian stock market, are all in different time zones with the U.S. stock market, which will result in the so-called non-synchroneity problem. Since the empirical study aims to demonstrate the advantages of copula-based local Kendall's tau relative to global Kendall's tau, we just simplify this problem by studying the co-movement relationship between S&P500 at time t and FTSE100, HSI, N225, SH at time t directly. We use ARMA(p,q)-GJR-GARCH(m,n)-Skewed student's t model to characterize all stock return series,[8] and then obtain standardized residual for each return series. We then use probability integral transformation to transform them to their corresponding empirical cumulative distribution function (ECDF) series, which are used for estimating the copula dependence structure between different stock returns.

As we know, the mixture copula proposed by Hu (2006) could capture more flexible dependence structure than a single copula (Eckernkemper 2017). Thus, we use Gumbel copula, rotated Gumbel copula (180 degrees), Clayton copula, and rotated Clayton copula (180 degrees) to construct four two-component bivariate mixture copulas to model the dependence structures between different stock return pairs,[9] which could capture several kinds of symmetric and asymmetric dependence structures. As mentioned in the previous section, it is hard to calculate local Kendall's tau of Gaussian copula and Student's t copula via formulas (7)(8)(9)(10) directly. Therefore, we do not consider these two copulas in the mixture copula model. These four mixture copulas are named as Mix1 copula, Mix2 copula, Mix3 copula and Mix4 copula respectively, which are expressed as

$$C_{Mix1}(u,v;\omega_G,\delta_G,\theta_C) = \omega_G C_{Gumbel}(u,v;\delta_G) + (1-\omega_G)C_{Clayton}(u,v;\theta_C)$$

[8] The detailed estimation results of ARMA(p,q)-GJR-GARCH(m,n)-Skewed student's t models are available from the authors upon request.

[9] In fact, we also consider several three-component mixture copulas which are constructed by Frank copula, Clayton copula and Gumbel copula. Although these three-component mixture copulas can improve the likelihood values, we find that the weights of Frank copula are all very small. Thus, following the suggestion of Cai and Wang (2014), Frank copula should not be included in the mixture copulas.

$$C_{Mix2}(u,v;\omega_G,\delta_{rG},\theta_{rC}) = \omega_{rG} C_{rotated\ Gumbel}(u,v;\delta_{rG}) + (1-\omega_{rG}) C_{rotated\ Clayton}(u,v;\theta_{rC})$$
$$C_{Mix3}(u,v;\omega_G,\delta_G,\theta_{rG}) = \omega_G C_{Gumbel}(u,v;\delta_G) + (1-\omega_G) C_{rotated\ Gumbel}(u,v;\theta_{rG})$$
$$C_{Mix4}(u,v;\omega_C,\theta_C,\theta_{rC}) = \omega_C C_{Clayton}(u,v;\theta_C) + (1-\omega_C) C_{rotated\ Clayton}(u,v;\theta_{rC})$$

where the rotated Gumbel copula (180 degrees) is the mirror image of Gumbel copula along the main diagonal. It is given by $C_{rotated\ Gumbel}(u,v;\delta_{rG}) = u + v - 1 + C_{Gumbel}(1-u, 1-v;\delta_{rG})$, $\delta_{rG} \in [1,+\infty]$; the rotated Clayton copula (180 degrees) is the mirror image of Clayton copula along the main diagonal. It is given by $C_{rotated\ Clayton}(u,v;\theta_{rC}) = u+v-1+C_{Clayton}(1-u,1-v;\theta_{rC})$, $\theta_{rC} \in [-1,0]\backslash\{0\}$. ω_G, ω_{rG}, ω_C are weight parameters of the mixture copulas, which all lie in interval $[0,1]$.

We use the maximum likelihood estimation (MLE) method to estimate above alternative mixture copulas for different stock return pairs, and then use the minimum Akaike Information Criteria (AIC) to select the best fitted copulas. Table 1 and 2 report the estimation results and the best fitted copulas for each stock return pair. The performance of the best fitted mixture copulas is evaluated by the χ^2 goodness-of-fit test (Hu, 2006). In the test, we divided the data into a table with $[8 \times 8]$ cells [10]. We merge cells whose expected frequency is 5 or less. We calculate the χ^2 statistic via expression: $M = \sum_{i=1}^{8}\sum_{j=1}^{8}\frac{(A_{ij}-B_{ij})^2}{B_{ij}}$. A_{ij} is the number of observed data in cell (i,j), B_{ij} is the theoretical frequency in cell (i,j) under the assumption that the dependence structure between two variables is the copula being tested. M is following χ^2 distribution with degree of freedom $(k-1)^2 - p - (q-1)$, where p represents the number of estimated parameters of different copulas, and q represents the number of cells that are merged together. The testing results in Table 3 indicate that our previous choices of the best copulas are appropriate.

Before using copula-based global and local Kendall's tau to study co-movement between U.S. and other international stock markets, we first calculate empirical global Kendall's tau of return pairs S&P500/S&P/TSX, S&P500/FTSE100, S&P500/HSI, S&P500/N225 and S&P500/SH with the empirical return data.[11] From Figure 5, we can see that return series S&P500 and S&P/TSX exhibit medium degree of global co-movement. S&P500 and FTSE100 exhibit weaker degree of global co-movement than S&P500/S&P/TSX, but it is not very low (around 0.38). The global dependence of return pairs S&P500/HSI, S&P500/N225 and S&P500/SH are relatively low, especially S&P500/SH. But we can not get more co-movement information from analyzing their empirical global Kendall's tau.

Next, we use empirical upper-upper and lower-lower local Kendall's tau along the main diagonal to further study co-movement between different stock markets. To ensure there exists enough data

[10] The number of cells is chosen by following Moore's rule (Moore 1986), which states that the reasonable number of cells is $2\sqrt[5]{n^2}$, where n is the number of observations. The number of observation in the current example is 4252. Thus, the number of column and row of the table are all $\sqrt[2]{2\sqrt[5]{n^2}} \approx 8$.

[11] The empirical return data refers to the ECDF series transformed from the standardized residual.

Table 1: Copula estimation results for stock return pairs S&P500/S&P/TSX, S&P500/FTSE100 and S&P500/HSI.

		S&P500/S&P/TSX		S&P500/FTSE100		S&P500/HSI	
		parameters	AIC	parameters	AIC	parameters	AIC
Mix1	ω_G	0.5262	-2906.12	0.6615	-1701.99	0.6369	**-267.32**
		(0.0292)		(0.0374)		(0.1212)	
	δ_G	2.0580		1.6106		1.1386	
		(0.0592)		(0.0393)		(0.0319)	
	θ_C	1.7040		1.1674		0.4474	
		(0.0989)		(0.1286)		(0.1668)	
Mix2	ω_{rG}	0.8181	-2921.75	0.6389	-1698.77	0.7503	-262.17
		(0.0248)		(0.0385)		(0.0756)	
	δ_{rG}	1.9258		1.6448		1.1289	
		(0.0336)		(0.0445)		(0.0194)	
	θ_{rC}	2.3932		1.0560		0.5925	
		(0.3132)		(0.1187)		(0.1904)	
Mix3	ω_G	0.3088	**-2934.75**	0.5143	**-1716.27**	0.4350	-265.04
		(0.0360)		(0.0482)		(0.2092)	
	δ_G	2.1377		1.5890		1.1925	
		(0.1166)		(0.0606)		(0.1203)	
	θ_{rG}	1.9149		1.6411		1.1482	
		(0.0460)		(0.0322)		(0.0712)	
Mix4	ω_C	0.6362	-2837.65	0.4867	-1663.88	0.6483	-263.76
		(0.0248)		(0.0810)		(0.1050)	
	θ_C	1.6882		1.1667		0.2745	
		(0.0712)		(0.0797)		(0.0555)	
	θ_{rC}	2.0334		1.0861		0.4711	
		(0.1483)		(0.0726)		(0.1551)	

Notes: The numbers in parentheses represent standard errors of the estimated parameters. The bold face numbers represent the minimum AIC values.

Table 2: Copula estimation results for stock return pairs S&P500/N225 and S&P500/SH.

		S&P500/N225		S&P500/SH	
		parameters	AIC	parameters	AIC
Mix1	ω_G	0.6169	-201.55	0.0171	**-17.71**
		(0.1257)		(0.0075)	
	δ_G	1.1020		10.2276	
		(0.0240)		(5.2550)	
	θ_C	0.4317		0.0410	
		(0.1448)		(0.0186)	
Mix2	ω_{rG}	0.6461	-202.92	0.9511	-15.83
		(0.1318)		(0.0924)	
	δ_{rG}	1.1470		1.0275	
		(0.0332)		(0.0098)	
	θ_{rC}	0.2714		0.5472	
		(0.0983)		(1.1839)	
Mix3	ω_G	0.4661	-201.20	0.9018	-16.53
		(0.1666)		(0.0831)	
	δ_G	1.1088		1.0082	
		(0.0436)		(0.0112)	
	θ_{rG}	1.1726		1.4298	
		(0.0601)		(0.5139)	
Mix4	ω_C	0.5013	**-204.79**	0.1162	-14.57
		(0.1395)		(0.0757)	
	θ_C	0.3505		0.5194	
		(0.1027)		(0.3461)	
	θ_{rC}	0.2380		0.0266	
		(0.0682)		(0.0191)	

Notes: The numbers in parentheses represent standard errors of the estimated parameters. The bold face numbers represent the minimum AIC values.

Table 3: The χ^2 goodness-of-fit test results of alternative copulas for each return pair.

Stock return pairs	Copulas	M statistic	Degree of freedom	Critical value (95%)
S&P500/S&P/TSX	Mix1 copula	58.0334*	45	61.6562
	Mix2 copula	56.0729*	45	61.6562
	Mix3 copula	**53.4794****	45	61.6562
	Mix4 copula	72.3124	45	61.6562
S&P500/FTSE100	Mix1 copula	56.9527*	47	64.0011
	Mix2 copula	61.0222*	47	64.0011
	Mix3 copula	**56.9451****	47	64.0011
	Mix4 copula	66.0804	47	64.0011
S&P500/HSI	**Mix1 copula**	**48.8250****	47	64.0011
	Mix2 copula	51.3576*	47	64.0011
	Mix3 copula	51.7287*	47	64.0011
	Mix4 copula	49.4447*	47	64.0011
S&P500/N225	Mix1 copula	53.1016*	47	64.0011
	Mix2 copula	57.8573*	47	64.0011
	Mix3 copula	55.9889*	47	64.0011
	Mix4 copula	**53.0283****	47	64.0011
S&P500/SH	**Mix1 copula**	**38.6959****	47	64.0011
	Mix2 copula	40.1749*	47	64.0011
	Mix3 copula	38.7006*	47	64.0011
	Mix4 copula	39.7579*	47	64.0011

Notes: Degrees of freedom in above table are calculated by $(k-1)^2 - p - (q-1)$ after merging cells whose expected frequency is 5 or less. * represents the acceptance of the null hypothesis at 95% confidence level, which means the data is well fitted by the marked copula. The best fitted copulas selected by AIC are represented in bold face.

for calculating empirical local Kendall's tau when two returns are very large or small, we restrict their quantiles in the interval [0.05,0.95]. Figure 5 shows that the empirical local Kendall's tau between different stock markets is not constant, but changing along the main diagonal. Moreover, empirical local Kendall's tau is obviously different from empirical global Kendall's tau, which implies that global Kendall's tau might overestimate or underestimate local Kendall's tau between different stock returns. Specifically, for return pairs S&P500/S&P/TSX and S&P500/FTSE100, their empirical local Kendall's tau at any quantile is obviously smaller than empirical global Kendall's tau. For the left three return pairs, empirical local Kendall's tau might be larger or smaller than the empirical global Kendall's tau.

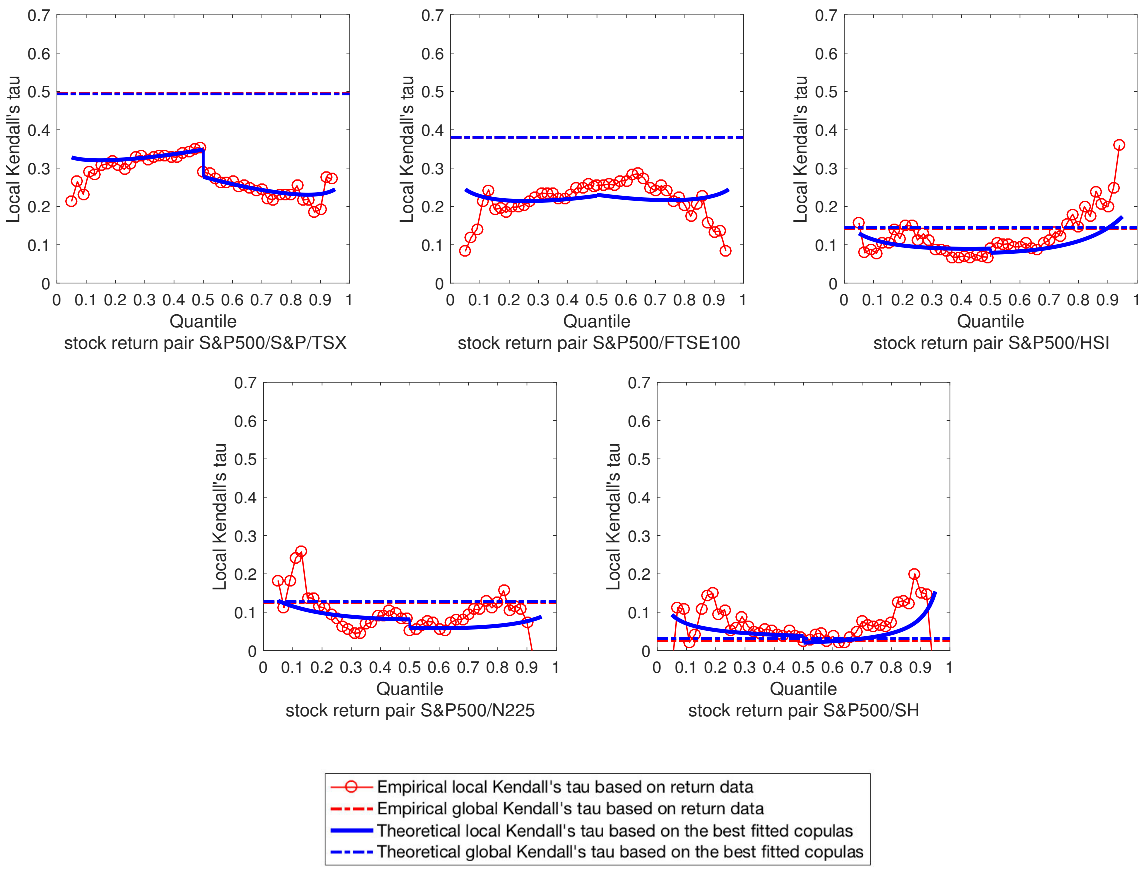

Figure 5. Global and Local Kendall's tau between different stock returns along the main diagonal. The red and blue dotted lines represent empirical and theoretical global Kendall's tau for different return pairs respectively. Note that the empirical global Kendall's tau and the theoretical global Kendall's tau are very close, thus these two lines are very close to each other. The blue smooth lines in intervals [0.05,0.5] and [0.5,0.95] represent theoretical $\tau_{LL}^{Kendall}$ and $\tau_{UU}^{Kendall}$ based on the best fitted copulas respectively. The non-smooth lines with circles represent empirical local Kendall's tau along the main diagonal calculated by the empirical return data.

Since we have already chosen the best fitted copulas for different stock return pairs, we use these copulas to calculate their theoretical global and local Kendall's tau via formulas (6) and

(7)(8)(9)(10) directly.[12] Then we draw the curves of their theoretical global and local Kendall's tau. We can observe from Figure 5 that, theoretical local Kendall's tau makes the characteristics of local dependence between different return series to be more clearer. Specifically, for S&P500/S&P/TSX, the theoretical local Kendall's tau is obviously smaller than the theoretical global Kendall's tau. The local dependence between S&P500 and S&P/TSX when both of them are in the boom situation is obvious smaller than their local dependence when both of them are in the crash situation, exhibiting an asymmetric local dependence pattern. In addition, their local dependence tends to be weaker as the increase or decrease of returns. Such tendency indicates that, S&P500 and S&P/TSX keep at a relatively high degree of co-movement when they are in normal situation, while the degree of co-movement between them becomes weaker as the increase and decrease of returns. We call it the "inversed V-type" co-movement pattern.

For S&P500/FTSE100, the theoretical local Kendall's tau is also obviously smaller than the theoretical global Kendall's tau, but the curve of the theoretical local Kendall's tau is very flat, implying that the degree of co-movement between S&P500 and FTSE100 does not change very much as the increase or decrease of returns.

For return pairs S&P500/HSI and S&P500/N225, they exhibit the similar co-movement pattern. Their theoretical local Kendall's tau are also smaller than theoretical global Kendall's tau in general. Compared with S&P500/S&P/TSX, the local co-movement of S&P500/HSI and S&P500/N225 tend to be stronger as the increase or decrease of returns, exhibiting a "V-type" co-movement pattern. However, they do not exhibit an obvious asymmetric characteristic.

Finally, the co-movement between S&P500 and SH seems very weak. From the global dependence perspective, their theoretical global Kendall's tau is 0.0312, indicating that there exist very low degree of dependence between S&P500 and SH. But we note that the global Kendall's tau underestimate the degree of local co-movement between them, because the theoretical and empirical local Kendall's tau are all larger than global Kendall's tau in general. Moreover, their values tend to be larger as the increase or decrease of returns, indicating that return pair S&P500/SH also exhibits a "V-type" co-movement pattern.

Next, we further study local dependence between U.S. and other five international stock markets when both of them are in adverse market conditions. We calculate their theoretical $\tau_{LL}^{Kendall}$ (along the main diagonal) at 5% and 10% quantiles respectively. Table 4 compares them with global Kendall's tau. We can see that, global Kendall's tau of S&P500/S&P/TSX and S&P500/FTSE100 are all larger than 0.35. However, their theoretical local Kendall's tau at 5% and 10% quantiles are all under 0.35, indicating that local dependence in the lower-lower tail regions for these two stock return pairs are not as strong as what global Kendall's tau shows to us. In addition, the theoretical

[12] As shown in the previous section, it is easy to draw the curves of the theoretical local Kendall's tau for Clayton copula and Gumbel copula via formulas (7-10) directly, and thus it is not difficult to draw the curves of different theoretical local Kendall's tau for a mixture copula that is constructed by the linear combination of Gumbel copula and Clayton copula. The Matlab codes for calculating the theoretical local Kendall's tau of different mixture copulas used in the empirical application are available from the authors upon request.

$\tau_{LL}^{Kendall}$ at 5% and 10% quantiles of S&P500/SH are all larger than their global Kendall's tau, implying that local dependence in lower-lower tail regions for S&P500/SH are not as weak as what global Kendall's tau shows to us.

Table 4: Comparison of local Kendall's tau and global Kendall's tau for different stock return pairs.

	Global Kendall's tau		Theoretical $\tau_{LL}^{Kendall}$	
Stock return pairs	Empirical	Theoretical	5%	10%
S&P500/S&P/TSX	0.4954	0.4935	0.3278	0.3215
S&P500/FTSE100	0.3798	0.3800	0.2472	0.2290
S&P500/HSI	0.1425	0.1441	0.1362	0.1174
S&P500/N225	0.1248	0.1281	0.1298	0.1154
S&P500/SH	0.0256	0.0311	0.0932	0.0703

In sum, we can get following main conclusions via above example: on the one hand, global Kendall's tau might overestimate and underestimate the degree of co-movement between two stock markets when both of them are in the boom or crash situations, on the other hand, copula-based local Kendall's tau can uncover richer local co-movement information than global Kendall's tau. These information could help us deepen the understanding of the relationship between two stock markets. Specifically, not only can copula-based local Kendall's tau characterize the symmetric and asymmetric co-movement patterns between different stock markets, but also the changing tendency of co-movement as the increase or decrease of stock returns.

5 Conclusion and Future Work

In this paper, we propose a novel copula-based local Kendall's tau framework, which could uncover richer nonlinear local dependence between two financial return series. The copula-based closed form formulas of different local Kendall's tau nest the copula-based formula of global Kendall's tau, providing a generalized framework for investigating dependence between two return series. We further extend the copula-based local dependence framework to Spearman's rho. We demonstrate the advantages of copula-based local Kendall's tau relative to global Kendall's tau with stock market data. The results indicate that, on the one hand, copula-based global Kendall's tau may overestimate or underestimate local dependence between two stock markets, on the other hand, copula-based local Kendall's tau can help us uncover richer dependence information than global Kendall's tau.

In the empirical study, we did not consider Gaussian copula and Student's t copula in the mixture copulas, because it is hard to calculate local Kendall's tau via formulas (7)(8)(9)(10) directly for these two copulas. Finding the simpler closed form solutions of local Kendall's tau for Gaussian and Student's t copulas (if there exists) is the direction of our future research. In addition, the copula-based local dependence framework can also be applied to Blomqvist's (1950) beta and Gini's Gamma (Nelsen 2006).

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Appendix A: Proof of theorem 1

Assuming that (X_1, Y_1) and (X_2, Y_2) are two independent realizations of (X, Y). According to expression (1), the population version of upper-upper local Kendall's tau is expressed by

$$\begin{aligned}\tau_{UU}^{Kendall}(X,Y;p,q) &= P\left((X_1-X_2)(Y_1-Y_2)>0 \mid X_1,X_2 \geq F_X^{-1}(p), Y_1,Y_2 \geq F_Y^{-1}(q)\right)\\ &\quad - P\left((X_1-X_2)(Y_1-Y_2)<0 \mid X_1,X_2 \geq F_X^{-1}(p), Y_1,Y_2 \geq F_Y^{-1}(q)\right)\end{aligned}$$

Since the random variables are continuous, we have

$$\begin{aligned}&P\left((X_1-X_2)(Y_1-Y_2)<0 \mid X_1,X_2 \geq F_X^{-1}(p), Y_1,Y_2 \geq F_Y^{-1}(q)\right)\\ &= 1 - P\left((X_1-X_2)(Y_1-Y_2)>0 \mid X_1,X_2 \geq F_X^{-1}(p), Y_1,Y_2 \geq F_Y^{-1}(q)\right)\end{aligned}$$

Therefore

$$\tau_{UU}^{Kendall}(X,Y;p,q) = 2P\left((X_1-X_2)(Y_1-Y_2)>0 \mid X_1,X_2 \geq F_X^{-1}(p), Y_1,Y_2 \geq F_Y^{-1}(q)\right) - 1$$

As we know

$$\begin{aligned}&P\left((X_1-X_2)(Y_1-Y_2)>0 \mid X_1,X_2 \geq F_X^{-1}(p), Y_1,Y_2 \geq F_Y^{-1}(q)\right)\\ &= \frac{P\left(X_1>X_2 \geq F_X^{-1}(p), Y_1>Y_2 \geq F_Y^{-1}(q)\right) + P\left(X_2>X_1 \geq F_X^{-1}(p), Y_2>Y_1 \geq F_Y^{-1}(q)\right)}{P\left(X_1,X_2 \geq F_X^{-1}(p), Y_1,Y_2 \geq F_Y^{-1}(q)\right)}\end{aligned}$$

We compute the probability of the above expression through integrating over the distribution of (X_1, Y_1) (Nelsen 2006, p.159) . First, the left part of the numerator can be expressed by

$$\begin{aligned}&P\left(X_1>X_2 \geq F_X^{-1}(p), Y_1>Y_2 \geq F_Y^{-1}(q)\right)\\ &= \int_{F_Y^{-1}(q)}^{+\infty}\int_{F_X^{-1}(p)}^{+\infty} P\left[F_X^{-1}(p) \leq X_2 < x, F_Y^{-1}(q) \leq Y_2 < y\right] dC_1(F_X(x),F_Y(y))\\ &= \int_{F_Y^{-1}(q)}^{+\infty}\int_{F_X^{-1}(p)}^{+\infty} \left[C_2(F_X(x),F_Y(y)) - C_2(p,F_Y(y)) - C_2(F_X(x),q) + C_2(p,q)\right] dC_1(F_X(x),F_Y(y))\end{aligned}$$

Similarly, the right part of the numerator can be expressed by

$$
\begin{aligned}
&P\left(X_2 > X_1 \geq F_X^{-1}(p), Y_2 > Y_1 \geq F_Y^{-1}(q)\right) \\
&= \int_{F_Y^{-1}(q)}^{+\infty} \int_{F_X^{-1}(p)}^{+\infty} P\left[F_X^{-1}(p) \leq x < X_2, F_Y^{-1}(q) \leq y < Y_2\right] dC_1(F_X(x), F_Y(y)) \\
&= \int_{F_Y^{-1}(q)}^{+\infty} \int_{F_X^{-1}(p)}^{+\infty} \left[1 - F_X(x) - F_Y(y) + C_2(F_X(x), F_Y(y))\right] dC_1(F_X(x), F_Y(y))
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
&P\left((X_1 - X_2)(Y_1 - Y_2) > 0 \mid X_1, X_2 \geq F_X^{-1}(p), Y_1, Y_2 \geq F_Y^{-1}(q)\right) \\
&= \frac{\int_{F_Y^{-1}(q)}^{+\infty} \int_{F_X^{-1}(p)}^{+\infty} \left[C_2(F_X(x), F_Y(y)) - C_2(p, F_Y(y)) - C_2(F_X(x), q) + C_2(p, q)\right] dC_1(F_X(x), F_Y(y))}{P\left(X_1, X_2 \geq F_X^{-1}(p), Y_1, Y_2 \geq F_Y^{-1}(q)\right)} \\
&\quad + \frac{\int_{F_Y^{-1}(q)}^{+\infty} \int_{F_X^{-1}(p)}^{+\infty} \left[1 - F_X(x) - F_Y(y) + C_2(F_X(x), F_Y(y))\right] dC_1(F_X(x), F_Y(y))}{P\left(X_1, X_2 \geq F_X^{-1}(p), Y_1, Y_2 \geq F_Y^{-1}(q)\right)}
\end{aligned}
$$

where $P\left(X_1, X_2 \geq F_X^{-1}(p), Y_1, Y_2 \geq F_Y^{-1}(q)\right) = (1 - p - q + C_1(p, q))(1 - p - q + C_2(p, q))$. Here, after applying probability integral transforms $U = F_X(x)$ and $V = F_Y(y)$, the above expression can be expressed by

$$
\begin{aligned}
&P\left((X_1 - X_2)(Y_1 - Y_2) > 0 \mid X_1, X_2 \geq F_X^{-1}(p), Y_1, Y_2 \geq F_Y^{-1}(q)\right) \\
&= \frac{\int_q^1 \int_p^1 \left[C_2(u, v) - C_2(p, v) - C_2(u, q) + C_2(p, q)\right] dC_1(u, v)}{(1 - p - q + C_1(p, q))(1 - p - q + C_2(p, q))} + \frac{\int_q^1 \int_p^1 \left[1 - u - v + C_2(u, v)\right] dC_1(u, v)}{(1 - p - q + C_1(p, q))(1 - p - q + C_2(p, q))}
\end{aligned}
$$

Since, $C_1 = C_2 = C$, and

$$
\tau_{UU}^{Kendall}(X, Y; p, q) = 2P\left((X_1 - X_2)(Y_1 - Y_2) > 0 \mid X_1, X_2 \geq F_X^{-1}(p), Y_1, Y_2 \geq F_Y^{-1}(q)\right) - 1
$$

Finally, upper-upper local Kendall's tau can be computed by following formula

$$
\tau_{UU}^{Kendall}(X, Y; p, q) = \frac{2\int_q^1 \int_p^1 [2C(u, v) - C(p, v) - C(u, q) - u - v] dC(u, v)}{(1 - p - q + C(p, q))^2} + \frac{1 + p + q + C(p, q)}{1 - p - q + C(p, q)}
$$

Similarly, we derive the copula-based formulas of other three local Kendall's tau

$$
\tau_{UL}^{Kendall}(X, Y; p, q) = \frac{2\int_0^q \int_p^1 [2C(u, v) - C(p, v) - C(u, q) - v] dC(u, v)}{(q - C(p, q))^2} + \frac{q + C(p, q)}{q - C(p, q)}
$$

$$\tau_{LU}^{Kendall}(X,Y;p,q) = \frac{2\int_q^1 \int_0^p [2C(u,v) - C(p,v) - C(u,q) - u] dC(u,v)}{(p - C(p,q))^2} + \frac{p + C(p,q)}{p - C(p,q)}$$

$$\tau_{LL}^{Kendall}(X,Y;p,q) = \frac{2\int_0^q \int_0^p [2C(u,v) - C(p,v) - C(u,q)] dC(u,v)}{C(p,q)^2} + 1$$

Since

$$\begin{aligned}
\int_0^q \int_p^1 C(u,q) dC(u,v) &= \int_0^q \int_p^1 C(u,q) \frac{\partial C(u,v)}{\partial u \partial v} du dv = \int_p^1 C(u,q) \big(\int_0^q \frac{\partial C(u,v)}{\partial u \partial v} dv \big) du \\
&= \int_p^1 C(u,q) \big(\frac{\partial C(u,q)}{\partial u} - \frac{\partial C(u,0)}{\partial u} \big) du = \int_p^1 C(u,q) \big(\frac{\partial C(u,q)}{\partial u} - 0 \big) du \\
&= \int_p^1 C(u,q) dC(u,q) = \frac{1}{2} \big(C(1,q)^2 - C(p,q)^2 \big) = \frac{1}{2} \big(q^2 - C(p,q)^2 \big) \\
\int_q^1 \int_0^p C(p,v) dC(u,v) &= \int_q^1 \int_0^p C(p,v) \frac{\partial C(u,v)}{\partial u \partial v} du dv = \int_q^1 C(p,v) \big(\int_0^p \frac{\partial C(u,v)}{\partial u \partial v} du \big) dv \\
&= \int_q^1 C(p,v) \big(\frac{\partial C(p,v)}{\partial v} - \frac{\partial C(0,v)}{\partial v} \big) dv \int_q^1 C(p,v) \big(\frac{\partial C(p,v)}{\partial v} - 0 \big) dv \\
&= \int_q^1 C(p,v) dC(p,v) = \frac{1}{2} \big(C(p,1)^2 - C(p,q)^2 \big) = \frac{1}{2} \big(p^2 - C(p,q)^2 \big)
\end{aligned}$$

Therefore, the formulas of $\tau_{UL}^{Kendall}(X,Y;p,q)$ and $\tau_{LU}^{Kendall}(X,Y;p,q)$ can be further simplified as

$$\tau_{UL}^{Kendall}(X,Y;p,q) = \frac{2\int_0^q \int_p^1 [2C(u,v) - C(p,v) - v] dC(u,v)}{(q - C(p,q))^2}$$

$$\tau_{LU}^{Kendall}(X,Y;p,q) = \frac{2\int_q^1 \int_0^p [2C(u,v) - C(u,q) - u] dC(u,v)}{(p - C(p,q))^2}$$

Similarly,

$$\begin{aligned}
\int_0^q \int_0^p C(p,v) dC(u,v) &= \int_0^q \int_0^p \big(C(p,v) \frac{\partial^2 C(u,v)}{\partial u \partial v} du \big) dv \\
&= \int_0^q C(p,v) \big(\int_0^p \frac{\partial^2 C(u,v)}{\partial u \partial v} du \big) dv \\
&= \int_0^q C(p,v) \big(\frac{\partial C(p,v)}{\partial v} - \frac{\partial C(0,v)}{\partial v} \big) dv \\
&= \int_0^q C(p,v) \frac{\partial C(p,v)}{\partial v} dv = \frac{1}{2} C(p,q)^2 = \int_0^q \int_0^p C(u,q) dC(u,v)
\end{aligned}$$

Therefore, the copula-based formula of $\tau_{LL}^{Kendall}(X,Y;p,q)$ can also be further simplified as

$$\tau_{LL}^{Kendall}(X,Y;p,q) = \frac{2\displaystyle\int_0^q\int_0^p \big[2C(u,v) - C(p,v) - C(u,q)\big]dC(u,v)}{C(p,q)^2} + 1$$
$$= \frac{2\displaystyle\int_0^q\int_0^p 2C(u,v)dC(u,v) - 2C(p,q)^2}{C(p,q)^2} + 1 = \frac{4\displaystyle\int_0^q\int_0^p C(u,v)dC(u,v)}{C(p,q)^2} - 1$$

The formulas in Theorem 1 have been proved.

Appendix B: Proof of Corollary 1

From formula (7), we know that

$$\lim_{p\to 0}\lim_{q\to 0}\tau_{UU}^{Kendall}(X,Y;p,q) = \frac{2\displaystyle\int_0^1\int_0^1 \big[2C(u,v) - C(0,v) - C(u,0) - u - v\big]dC(u,v)}{(1+C(0,0))^2} + \frac{1+C(0,0)}{1+C(0,0)}$$
$$= \frac{2\displaystyle\int_0^1\int_0^1 2C(u,v)dC(u,v) - 2\int_0^1\int_0^1 udC(u,v) - 2\int_0^1\int_0^1 vdC(u,v)}{1} + 1$$

Since $E(U) = E(V) = 1/2$ (Nelsen 2006, p.160), we have $\displaystyle\int_0^1\int_0^1 udC(u,v) = \int_0^1\int_0^1 vdC(u,v) = \frac{1}{2}$. Therefore,

$$\lim_{p\to 0}\lim_{q\to 0}\tau_{UU}^{Kendall}(X,Y;p,q) = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1$$

Appendix C: The proof of equations (11)(12) and (13)

As we know, $F_{-X}^{-1}(p) = -F_X^{-1}(1-p)$, therefore

$$\begin{aligned}\tau_{LL}^{Kendall}(-X,Y \mid -X \le F_{-X}^{-1}(1-p), Y \le F_Y^{-1}(q)) &= \tau_{LL}^{Kendall}(-X,Y \mid -X \le -F_X^{-1}(p), Y \le F_Y^{-1}(q))\\ &= \tau_{UL}^{Kendall}(-X,Y \mid X \ge F_X^{-1}(p), Y \le F_Y^{-1}(q))\\ &= -\tau_{UL}^{Kendall}(X,Y \mid X \ge F_X^{-1}(p), Y \le F_Y^{-1}(q))\end{aligned}$$

which proves the equation (11), $\tau_{UL}^{Kendall}(X,Y;p,q) = -\tau_{LL}^{Kendall}(-X,Y;1-p,q)$. Similarly, we can prove the equations (12) and (13).

Appendix D: Proof of Theorem 2

Assuming that (X_1, Y_1), (X_2, Y_2) and (X_3, Y_3) are three independence realizations of (X, Y). Therefore, X_1 and Y_1 are joint by copula function C, while X_2 and Y_3 are joint by independence copula function $C_\perp$. From Nelsen (2006, p. 167), we know that the expression of $\rho_{UU}^{Spearman}(X, Y; p, q)$ can be expressed by

$$\begin{aligned}\rho_{UU}^{Spearman}(X, Y; p, q) &= 3P\left((X_1 - X_2)(Y_1 - Y_3) > 0 \mid X_1, X_2 \geq F_X^{-1}(p), Y_1, Y_3 \geq F_Y^{-1}(q)\right) \\ &\quad - 3P\left((X_1 - X_2)(Y_1 - Y_3) < 0 \mid X_1, X_2 \geq F_X^{-1}(p), Y_1, Y_3 \geq F_Y^{-1}(q)\right) \\ &= 6P\left((X_1 - X_2)(Y_1 - Y_3) > 0 \mid X_1, X_2 \geq F_X^{-1}(p), Y_1, Y_3 \geq F_Y^{-1}(q)\right) - 3\end{aligned}$$

We know

$$\begin{aligned}&P\left((X_1 - X_2)(Y_1 - Y_3) > 0 \mid X_1, X_2 \geq F_X^{-1}(p), Y_1, Y_2 \geq F_Y^{-1}(q)\right) \\ &= \frac{P\left(X_1 > X_2 \geq F_X^{-1}(p), Y_1 > Y_3 \geq F_Y^{-1}(q)\right) + P\left(X_2 > X_1 > F_X^{-1}(p), Y_3 > Y_1 > F_Y^{-1}(q)\right)}{P\left(X_1, X_2 \geq F_X^{-1}(p), Y_1, Y_3 \geq F_Y^{-1}(q)\right)}\end{aligned}$$

where $P\left(X_1, X_2 \geq F_X^{-1}(p), Y_1, Y_3 \geq F_Y^{-1}(q)\right) = (1 - p - q + C(p, q))(1 - p - q + pq)$.
As we know

$$\begin{aligned}&P\left(X_1 > X_2 \geq F_X^{-1}(p), Y_1 > Y_3 \geq F_Y^{-1}(q)\right) \\ &= \int_{F_Y^{-1}(q)}^{+\infty} \int_{F_X^{-1}(p)}^{+\infty} P\left(F_X^{-1}(p) \leq X_2 < x, F_Y^{-1}(q) \leq Y_3 < y\right) dC(F_X(x), F_Y(y)) \\ &= \int_{F_Y^{-1}(q)}^{+\infty} \int_{F_X^{-1}(p)}^{+\infty} \left[C_\perp(F_X(x), F_Y(y)) - C_\perp(p, F_Y(y)) - C_\perp(F_X(x), q) + C_\perp(p, q)\right] dC(F_X(x), F_Y(y)) \\ &= \int_{F_Y^{-1}(q)}^{+\infty} \int_{F_X^{-1}(p)}^{+\infty} \left[F_X(x)F_Y(y) - pF_Y(y) - F_X(x)q + pq\right] dC(F_X(x), F_Y(y))\end{aligned}$$

Similarly, we have

$$\begin{aligned}&P\left(X_2 > X_1 \geq F_X^{-1}(p), Y_3 > Y_1 \geq F_Y^{-1}(q)\right) \\ &= \int_{F_Y^{-1}(q)}^{+\infty} \int_{F_X^{-1}(p)}^{+\infty} P\left(F_X^{-1}(p) \leq x < X_2, F_Y^{-1}(q) \leq y < Y_3\right) dC(F_X(x), F_Y(y)) \\ &= \int_{F_Y^{-1}(q)}^{+\infty} \int_{F_X^{-1}(p)}^{+\infty} \left[1 - F_X(x) - F_Y(y) + C_\perp(F_X(x), F_Y(y))\right] dC(F_X(x), F_Y(y))\end{aligned}$$

From the above calculation procedure, we have

$$\begin{aligned}&P\left((X_1 - X_2)(Y_1 - Y_3) > 0 \mid X_1, X_2 \geq F_X^{-1}(p), Y_1, Y_2 \geq F_Y^{-1}(q)\right) \\ &= \frac{\displaystyle\int_{F_Y^{-1}(q)}^{+\infty} \int_{F_X^{-1}(p)}^{+\infty} \left[F_X(x)F_Y(y) - pF_Y(y) - F_X(x)q + pq\right] dC(F_X(x), F_Y(y))}{(1 - p - q + C(p, q))(1 - p - q + pq)}\end{aligned}$$

$$+\frac{\displaystyle\int_{F_Y^{-1}(q)}^{+\infty}\int_{F_X^{-1}(p)}^{+\infty}\left[1-F_X(x)-F_Y(y)+C_\perp(F_X(x),F_Y(y))\right]dC(F_X(x),F_Y(y))}{(1-p-q+C(p,q))(1-p-q+pq)}$$

After applying probability integral transform, we finally get the expression of $\rho_{UU}^{Spearman}$

$$\rho_{UU}^{Spearman}(X,Y;p,q)=\frac{6\displaystyle\int_q^1\int_p^1[2uv-(p+1)v-(q+1)u+pq+1]dC(u,v)}{(1-p-q+C(p,q))(1-p-q+pq)}-3$$

Similarly, we calculate the formulas of the other three local Spearman's rho respectively

$$\rho_{UL}^{Spearman}(X,Y;p,q)=\frac{6\displaystyle\int_0^q\int_p^1[2uv-qu-(p+1)v+q]dC(u,v)}{q(q-C(p,q))(1-p)}-3$$

$$\rho_{LU}^{Spearman}(X,Y;p,q)=\frac{6\displaystyle\int_q^1\int_0^p[2uv-(q+1)u-pv+p]dC(u,v)}{p(p-C(p,q))(1-q)}-3$$

$$\rho_{LL}^{Spearman}(X,Y;p,q)=\frac{6\displaystyle\int_0^q\int_0^p[2uv-qu-pv]dC(u,v)}{pqC(p,q)}+3$$